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Convergence Classes and Spaces of Partial Functions

Anthony K. Seda

Roland Heinze

Pascal Hitzler
pascal.hitzler@wright.edu

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Roland Heinze
Institut für Informatik III
Rheinische Friedrich-Wilhelms-Universität Bonn
Römerstr. 164, 53117 Bonn, Germany

Pascal Hitzler†
Artificial Intelligence Institute, Dresden University of Technology
01062 Dresden, Germany

Anthony Karel Seda
Department of Mathematics, University College Cork
Cork, Ireland

Abstract

We study the relationship between convergence spaces and convergence classes given by means of both nets and filters, we consider the duality between them and we identify in convergence terms when a convergence space coincides with a convergence class. We examine the basic operators in the Vienna Development Method of formal systems development, namely, extension, glueing, restriction, removal and override, from the perspective of the Logic for Computable Functions. Thus, we examine in detail the Scott continuity, or otherwise, of these operators when viewed as operators on the domain $(X \to Y)$ of partial functions mapping X into Y. The important override operator is not Scott continuous, and we consider topologies defined by convergence classes which rectify this situation.

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1 Introduction

The uses of topology in studying theoretical aspects of computer science are varied and wide. Many of them are related to domain theory and to programming language semantics and hence, ultimately, to the Scott topology. But other important applications are known, including: digital topology in image processing, the use of ideas from homotopy theory in "no deadlock" proofs in concurrency, and the use of topology in logic programming, to mention a few.

Of course, there are also many ways of specifying topologies, varying from the assignment of families of sets to be taken as open, to the use of (ultra)metrics of various generality, through to order-theoretic means. Yet another familiar method, though not, it appears, widely used in computing is to specify convergence by means of convergence classes. Here, one is concerned with conditions on a family of pairs consisting of a net in a set together with a point of that set so that the given family generates a topology in which the convergent nets (and their limits) are precisely the members of the given class. As a matter of fact, the Scott topology on a domain has a simple characterization in these terms and we will use this later.

On the other hand, convergence structures (known as convergence spaces) more general than topological spaces have been investigated in [1], and elsewhere, as a means of unifying discrete and continuous models of computation, or hybrid systems. In a convergence space, one is specifically given a notion of convergence at each point by means of families of filters, see Section 2 and [1, 3, 5] for details. The notion of convergence thus specified is much weaker than that which prevails in a topological space, although each convergence space has a topology naturally associated with it. Furthermore, the topology just mentioned has, in turn, an associated convergence class, and herein we call a convergence space *topological* if it coincides with its associated convergence class. The embedding of topological spaces into convergence spaces implied thereby is, of course, strict, see [3].

The motivation for this paper is the examination of certain aspects of convergence in computer science, and is threefold, as follows.

First, it is precisely the convergence properties of the topologies used in [4, 12] which are most useful in relation to termination and semantical questions in logic programming in the presence of negation. Whilst not originally defined by means of convergence, the convergence properties of these topologies are natural and could indeed have been taken as definitive. This point is examined in some detail in [13] where it is shown that convergence can be taken as a fundamental concept in unifying the procedural point of view and the declarative point of view in the context of logic programming with negation.

The second point concerns the Vienna Development Method (VDM) of formal system specification as expounded particularly in [6, 9, 10]. VDM is a development method which starts with the formal specification of the system requirements and ends, after a sequence of refinement steps, with the implemented program code. At each refinement step, a number of proof obligations have to be fulfilled which ensure that system requirements are met. In the form of VDM developed in [9, 10], denoted by VDM$^\clubsuit$ and termed the Irish School of the VDM, preconditions are used, but not postconditions. Instead of using formal logic to verify postconditions, proof obligation (of system invariants) is carried out constructively using a calculus of operators defined on spaces of partial functions. The calculus aims, of course, at reducing complicated calculations to routine symbol manipulation, especially those calculations concerned with things like domain restriction and removal, extension of functions and, in particular, override of functions (which is an important tool in modelling the process of updating records, file systems etc.).

On the other hand, spaces of partial functions, and operators defined on them, arise as particularly important examples of domains in Scott's well-established, and extensive, Logic for Computable Functions (LCF), see [11], which formalizes an abstract model of computability. Thus, although their aims are rather different, it is of interest to contrast VDM$^\clubsuit$ and LCF to the extent of investigating the operators which arise within VDM$^\clubsuit$ from the point of view of LCF, and specifically to determine their computability, or otherwise, in terms of Scott-continuity, and it is the second main purpose of this article to take some initial steps in carrying out this process. Thus, we intend to study in detail the basic operators arising in VDM$^\clubsuit$ when considered as operators on the domain $(X \to Y)$ of partial functions mapping X to Y, and to examine their continuity principally in relation to the Scott topology. However, it turns out that one of the most important operators, the override, is not Scott continuous, and this fact necessitates the introduction of other topologies, related to the Scott topology, to describe its behaviour. The particular topology we introduce here is in fact the smallest refinement of the Scott and Lawson topologies meeting certain natural conditions, see Proposition 3.18. All this is done by means of convergence classes, and we obtain thereby a simple and natural treatment.

Third, we want to examine more closely the relationship between convergence spaces and topological spaces from the point of view of convergence in an attempt to better understand convergence spaces and their applications to modelling hybrid systems, including applications to spaces of valuations in logic programming, and to spaces of partial functions as in Section 3. Indeed, this paper and [13] are complementary: in [13] spaces of valuations in many-valued logics are considered from the point of view of convergence, and here we focus on spaces of partial functions from the same point of view.

In effect, the paper falls naturally into two parts. In the first of these, Section 2, we present a definition of convergence spaces in terms of nets, and a definition of convergence classes in terms of filters, both of which are new. Once that is done, the hoped-for duality between convergence spaces and convergence classes in filter form, on the one hand, and convergence spaces and convergence classes in net form, on the other, can be and, indeed, is established, in Section 2; it is of course derived from the usual duality between nets and filters. Moreover, we view convergence spaces as generalizations of convergence classes (topological convergence spaces), and we are able to identify precise conditions, in terms of convergence, under which a convergence space is topological. In addition, there is the usual advantage of having both formulations available: nets are intuitive and easy to use to check, say, continuity; filters are preferred when features of the space need to be involved.

The second part of the paper, Sections 3 and 4, are devoted to the study of the operators arising in VDM by means of convergence classes, as already mentioned. Taken together with [13], it gives a detailed treatment, based on the convergence concepts in the first part, of two of the main structures encountered in the theory of computation: spaces of valuations and spaces of partial functions. Moreover, it addresses the question, by analogy with areas of mathematical analysis, of what is a reasonable notion of convergence in spaces of partial functions. This question was in fact one of the original motivations for the paper, and the answers we provide are, we believe, both elegant and interesting quite apart from any applications to VDM.

Acknowledgement We thank an anonymous referee for making some suggestions which clarified certain results in the paper, and for drawing our attention to the need to examine the effectiveness of the operators we discuss. This latter point is something which is already under consideration by the authors in relation to the work of [6, 9, 10] and will be treated in detail elsewhere, but see the remarks made in Section 5, Conclusions and Further Work.

2 Convergence Spaces and Convergence Classes

2.1 Preliminaries

We assume that the reader is familiar with the basic properties of nets, and we use [7, 16] as our general references to this topic and for much of our notation. Thus, a *net* in a set X is a function $S : D \to X$, where $(D, \leq)$ is some directed set. The point $S(n), n \in D$, is often denoted S_n or x_n and we frequently refer to "the net $(S_n)_{n\in D}$" or "the net $(x_n)_{n\in D}$" instead of the net S. If no confusion can arise, we use (S_n) as an abbreviation for $(S_n)_{n\in D}$. A *tail* of a net (x_n) in X is a set of the form $\{x_n \,|\, n \geq m\}$, where m is an element of D. As usual, if (x_n) is a net in X, then a property will be said to hold *eventually* with respect to (x_n) if it holds for all $n \geq m$ for some element m of the index set of (x_n).

One point on which we will be specific, however, is in our use of the term "subnet", and we will adopt Kelley's definition throughout (see [7]), noting that this form is more general than that employed in [16]. Thus, a *subnet* T of a net $S : D \to X$ is a net $T : M \to X$ satisfying: (i) $T = S \circ \varphi$, where φ is a function mapping M into D, and (ii) for each $n \in D$, there exists $m \in M$ such that $\varphi(p) \geq n$ whenever $p \geq m$. The point $S \circ \varphi\,(m)$ is often denoted S_{n_m} or x_{n_m}, and we refer to the subnet $(x_{n_m})_{m\in M}$ of $(x_n)_{n\in D}$.

As usual, a net (S_n) in a topological space X will be said to *converge* to $x \in X$, written $S_n \to x$ or $\lim_n S_n = x$, if each neighbourhood U of x contains a tail of (S_n). The following elementary fact will be used quite often in the sequel: *if E is a subset of a topological space X, then $x \in \overline{E}$ iff there exists a net (x_n) in E with $x_n \to x$*, where $\overline{E}$ denotes the closure of E in X.

Concerning filters, filter bases and ultrafilters, we again assume a basic familiarity with these topics and refer the reader to [16] for all the background and notation we need. Thus, a filter $\mathcal{A}$ on a set X is a non-empty collection of non-empty sets closed under the processes of taking finite intersections and superset. In particular, a filter $\mathcal{A}$ in a topological space X will be said to *converge* to $x \in X$, written $\mathcal{A} \to x$, if $\mathcal{A}$ is finer than the neighbourhood filter $N(x)$ at x, that is, $N(x) \subseteq \mathcal{A}$. Also, if $A \subseteq X$, then the filter determined by A, namely the set of all supersets of A, will be denoted by $[A]$; in case A is a singleton set $\{x\}$, we denote this ultrafilter by $[x]$ and refer to it as the *point filter at* x. The analogue in terms of filters of the earlier-mentioned elementary fact concerning closure, which again will be used quite often, is the following: *if E is a subset of a topological space X, then $x \in \overline{E}$ iff there exists a filter $\mathcal{A}$ on X with $E \in \mathcal{A}$ and $\mathcal{A} \to x$.*

Of course, the theory of nets and the theory of filters are dual in that any topological fact that can be established by means of the one can equally well be established by means of the other. The exact translation of each of these theories into the other can be found in many places, but we follow [16] in this regard and include next the bare details which we will need later.

Let (x_n) be a net in X and, for each $n \in D$, let $B_n = \{x_m \,|\, m \geq n\}$ be a tail of (x_n). Let $\mathcal{C}$ denote the collection $\{B_n \,|\, n \in D\}$. Then $\mathcal{C}$ is the base for a filter called the *filter generated by the net* (x_n). On the other hand, let $\mathcal{A}$ be a filter on X and let $D_{\mathcal{A}}$ denote the set $\{(x, A) \,|\, x \in A \in \mathcal{A}\}$. We define an ordering $\leq$ on $D_{\mathcal{A}}$ by $(x_1, A_1) \leq (x_2, A_2)$ if $A_2 \subseteq A_1$. Then $(D_{\mathcal{A}}, \leq)$ is a directed set, and the mapping $S\colon D_{\mathcal{A}} \to X$ defined by $S(x, A) = x$ determines a net in X called the *net based on the filter* $\mathcal{A}$. The precise connection between these two notions is contained in the following result.

2.1 Theorem Let X be a topological space and let x be an element of X. Then the following statements hold.
(a) A filter $\mathcal{A}$ on X converges to x iff the net based on $\mathcal{A}$ converges to x.
(b) A net (x_n) in X converges to x iff the filter generated by (x_n) converges to x.

Finally, we remind the reader that a closure operator (also known as a Kuratowski, or topological, closure operator) on a set X is a mapping $^c : \mathcal{P}(X) \to \mathcal{P}(X)$, from the power set $\mathcal{P}(X)$ of X into itself, subject to the following axioms. (a) $\emptyset^c = \emptyset$. (b) $A \subseteq A^c$ for all $A \subseteq X$. (c) $(A \cup B)^c = A^c \cup B^c$ for all $A, B \subseteq X$. (d) $A^c = (A^c)^c$ for all $A \subseteq X$.

The main fact we need concerning closure operators is the following well-known theorem. *Let X be a non-empty set and let $^c : \mathcal{P}(X) \to \mathcal{P}(X)$ be a closure operator on X. Then $\mathcal{T} = \{X \setminus A \,|\, A \subseteq X, A = A^c\}$ is a topology on X in which $\overline{A} = A^c$ for each subset A of X. Thus, A^c is the topological closure in X of each subset A of X with respect to the topology $\mathcal{T}$ determined by ^{c}.*

2.2 Convergence Spaces in Filter and Net Form

We expound here only that part of the theory of convergence spaces which is closely related to the theory of convergence classes as needed later on. For more information on convergence spaces given in terms of filters, see [1, 3, 5].

We begin this section by studying convergence spaces given in the conventional way in terms of filters. We shall henceforth usually refer to these as “convergence spaces in filter form” in order to distinguish them from the form we give later using nets, which is new. The three results we present concerning convergence spaces in filter form are well-known, see [1], but we include those details of proof, of the third, that we need later on.

2.2 Definition (Convergence Space in Filter Form) Let X be a non-empty set. The pair $(X, \mathcal{F}) = (X, (\mathcal{F}_x)_{x \in X})$ is called a *convergence space in filter form* if, for each $x \in X$, $\mathcal{F}_x$ is a collection of filters on X with the following properties.

(a) The point filter $[x]$ belongs to $\mathcal{F}_x$ for each $x \in X$ (point filter axiom).

(b) If $\mathcal{A} \in \mathcal{F}_x$ and $\mathcal{B} \supseteq \mathcal{A}$ is a filter on X, then $\mathcal{B} \in \mathcal{F}_x$ (closure under superfilters).

One sometimes uses another notation for convergence spaces in filter form. One writes $\mathcal{A} \downarrow x$ iff $\mathcal{A} \in \mathcal{F}_x$ and refers to the convergence space $(X, \downarrow)$ instead of $(X, \mathcal{F})$. Thus, we interpret $\downarrow$ as a relation between filters on X and elements of X. If $\mathcal{A} \in \mathcal{F}_x$, we say $\mathcal{A}$ *converges to* x.

We say that a convergence space $(X, \mathcal{F})$ in filter form is *pointed* if, for each $x \in X$, we have $\bigcap \mathcal{F}_x \in \mathcal{F}_x$.

Finally, a subset $O \subseteq X$ is said to be *open* in the *induced topology* of a convergence space $(X, \mathcal{F})$ in filter form if $\mathcal{A} \downarrow x \in O$ always implies $O \in \mathcal{A}$ (so that $\mathcal{A} \supseteq [O]$), where $\mathcal{A} \downarrow x \in O$ means "$\mathcal{A}$ converges to x and $x \in O$".

2.3 Lemma For each $x \in X$, $\bigcap \mathcal{F}_x$ is a filter coarser than each $\mathcal{A} \in \mathcal{F}_x$. Furthermore, for each $A \in \bigcap \mathcal{F}_x$, we have $x \in A$.

2.4 Lemma The induced topology of a convergence space $(X, \mathcal{F})$ in filter form is a topology on X.

2.5 Lemma Every topological space $(X, \mathcal{T})$ is representable as a convergence space $(X, \mathcal{F})$ in filter form such that the induced topology of $(X, \mathcal{F})$ is $\mathcal{T}$.

Proof. For each $x \in X$, let $\mathcal{F}_x$ be the set of all filters $\mathcal{A}$ on X such that $\mathcal{A} \supseteq N(x)$, where $N(x)$ denotes the neighbourhood filter at x with respect to the topology $\mathcal{T}$. Obviously, $\mathcal{F}_x$ has the properties listed in the definition of a convergence space in filter form. So, we choose $\mathcal{F} = (\mathcal{F}_x)_{x \in X}$.

First, let $O \in \mathcal{T}$ and suppose that $\mathcal{A} \downarrow x \in O$. We immediately get $O \in N(x) \subseteq \mathcal{A}$ and, as x was chosen arbitrarily, we see that O is open with respect to the induced topology.

Next, let O be open with respect to the induced topology. Then $N(x) \downarrow x \in O$ implies $O \in N(x)$. Therefore, there exists an open set $O_x \in N(x) \cap \mathcal{T}$ with $O_x \subseteq O$, and we obtain $O = \bigcup_{x \in O} O_x \in \mathcal{T}$. So, O is open with respect to $\mathcal{T}$, as required. ■

2.6 Definition Let $(X, \mathcal{T})$ be a topological space. The *convergence space in filter form induced by* $\mathcal{T}$ is defined as follows: $\mathcal{A} \downarrow_{\mathcal{O}} x$ iff $\mathcal{A} \supseteq N(x)$ is a filter on X, where $N(x)$ is the neighbourhood filter at x in $\mathcal{T}$. A convergence space $(X, \downarrow)$ in filter form is called a *topological convergence space (in filter form)* if there is a topology $\mathcal{T}$ on X with the property that the convergence space in filter form induced by $\mathcal{T}$ coincides with $(X, \downarrow)$. We sometimes use the notation $(X, \downarrow_{\mathcal{O}})$ to indicate that a convergence space is a topological convergence space.

We now turn our attention to formulating the notion of convergence space in terms of nets, and obtain duals of each of the results above. Results of the later parts of this section show that the definition we give and the corresponding duality are both highly satisfactory.

2.7 Definition (Convergence Space in Net Form) Let X be a non-empty set. The pair $(X, \mathcal{S}) = (X, (\mathcal{S}_s)_{s \in X})$ is called a *convergence space in net form* if, for each $s \in X$, $\mathcal{S}_s$ is a non-empty collection of nets in X with the following properties.

(a) If $S\colon D \to X$ is a constant net, that is, $S_n = s \in X$ for all $n \in D$, then $S \in \mathcal{S}_s$.

(b) If $S \in \mathcal{S}_s$ and T is a subnet of S, then $T \in \mathcal{S}_s$.

We will sometimes use another notation for convergence spaces in net form. We write $S \downarrow s$ iff $S \in \mathcal{S}_s$ and refer to the convergence space $(X, \downarrow)$ instead of $(X, \mathcal{S})$. Thus, we interpret $\downarrow$ as a relation between nets in X and elements of X. If $S \in \mathcal{S}_s$, we say S *converges to* s.

A subset $O \subseteq X$ is said to be *open* in the *induced topology* of a convergence space $(X, \mathcal{S})$ in net form if $S \downarrow s \in O$ always implies that there exists $n \in D$ with $S_m \in O$ for all $m \geq n$. Here, of course, $S \downarrow s \in O$ means "S converges to s and $s \in O$".

By analogy with the results above for convergence spaces in filter form, we prove that the induced topology just defined really is a topology, and that every topological space is representable as a convergence space in net form in such a way that the induced topology of the convergence space coincides with the original topology of the topological space.

2.8 Lemma The induced topology of a convergence space $(X, \mathcal{S})$ in net form is a topology on X.

Proof. From the definition, $\emptyset$ and X are obviously open sets in the induced topology.

Let $(O_i)_{i \in I}$ be a family of open sets in the induced topology of $(X, \mathcal{S})$. Let $O = \bigcup_{i \in I} O_i$, and suppose that $S \downarrow s \in O$. Then there exists $i \in I$ and $n \in D$ such that $s \in O_i$ and $S_m \in O_i \subseteq O$ for all $m \geq n$. Since $s \in O$ was chosen arbitrarily, we see that O is an element of the induced topology.

Now let $O_1, O_2 \subseteq X$ be open sets in the induced topology of $(X, \mathcal{S})$. Let $O = O_1 \cap O_2$ and suppose that $S \downarrow s \in O$. Then there exist $n_1, n_2 \in D$ such that $S_n \in O_i$ for all $n \geq n_i, i = 1, 2$. Since D is directed, there exists $n_0 \in D$ with $n_0 \geq n_1, n_2$, and we have $S_n \in O$ for all $n \geq n_0$. Because s is an arbitrary element of O, we conclude that O is an element of the induced topology, as required. ■

2.9 Lemma Every topological space $(X, \mathcal{T})$ is representable as a convergence space $(X, \mathcal{S})$ in net form such that the induced topology of $(X, \mathcal{S})$ is $\mathcal{T}$.

Proof. We define $S \in \mathcal{S}_s$ iff $S : D \to X$ is a net with $S_n \to s$ with respect to $\mathcal{T}$. Obviously $\mathcal{S}_s, s \in X$, fulfills the conditions listed in the definition of a convergence space in net form. So, we choose $\mathcal{S} = (\mathcal{S}_s)_{s \in X}$.

We show that the induced topology of $(X, \mathcal{S})$ is $\mathcal{T}$. First, let O be open with respect to $\mathcal{T}$ and suppose that $S \downarrow s \in O$. By definition, we have $S_n \to s$ with respect to $\mathcal{T}$ and, because $O \in \mathcal{T}$, we conclude that there exists $n \in D$ with $S_m \in O$ for all $m \geq n$. Since s is an arbitrary element of O, we see that O is an open set of the induced topology of $(X, \mathcal{S})$.

Next, let O be open with respect to the induced topology. Suppose $O \notin \mathcal{T}$ so that $X \setminus O$ is not closed with respect to $\mathcal{T}$. Then there exists $s \in \overline{X \setminus O}^{\mathcal{T}} \cap O$, where $\overline{X \setminus O}^{\mathcal{T}}$ denotes the closure of $X \setminus O$ in X relative to $\mathcal{T}$. Thus, by the elementary facts noted earlier, there exists a net $S : D \to X$ with $S_n \in X \setminus O$ for all $n \in D$, and also $S_n \to s \in O$. Hence, we have $S \downarrow s \in O$. But O is open in the induced topology. Therefore, there exists $n \in D$ such that $S_m \in O$ for all $m \geq n$, which contradicts the fact that $S_n \in X \setminus O$ for all $n \in D$. So, we conclude that $O \in \mathcal{T}$, as required. ■

2.10 Definition Let $(X, \mathcal{T})$ be a topological space. The *convergence space in net form induced by* $\mathcal{T}$ is defined as follows: $S \downarrow_{\mathcal{O}} s$ iff S is a net in X with $S_n \to s$ with respect to $\mathcal{T}$. A convergence space $(X, \downarrow)$ in net form is called a *topological convergence space (in net form)* if there is a topology $\mathcal{T}$ on X with the property that the convergence space in net form induced by $\mathcal{T}$ coincides with $(X, \downarrow)$. We sometimes use the notation $(X, \downarrow_{\mathcal{O}})$ to indicate that a convergence space is a topological convergence space.

2.3 Convergence Classes in Net and Filter Form

As already noted, convergence spaces are normally defined in terms of filters whilst convergence classes are defined in terms of nets. We begin this section by briefly considering convergence classes defined by means of nets, following [7], before presenting a treatment of them defined by means of filters which gives the duality we want between the two approaches. The same terminology (net and filter form) as used in the previous section will be adopted here and in the sequel to distinguish the two definitions.

2.11 Definition (Convergence Class in Net Form) Let X be an arbitrary non-empty set. We call $\mathcal{C}$ a *convergence class for X in net form* if $\mathcal{C}$ is a set of pairs each consisting of a net S in X and a point s of X such that the conditions listed below are satisfied. Instead of $(S, s) \in \mathcal{C}$ we also use the notation S *converges* $(\mathcal{C})$ *to* s or $\lim_n S_n \equiv s\,(\mathcal{C})$, see [7, Page 74].

(a) If $S\colon D \to X$ is a constant net in X, that is, $S_n = s \in X$ for all $n \in D$, then $(S, s) \in \mathcal{C}$.

(b) If $(S, s) \in \mathcal{C}$ and T is a subnet of S, then $(T, s) \in \mathcal{C}$.

(c) If $(S, s) \notin \mathcal{C}$, then there exists a subnet T of S such that for every subnet R of T we have $(R, s) \notin \mathcal{C}$.

(d) Let D be a directed set, let E_m be a directed set for each $m \in D$, let F denote the product directed set $D \times \prod_{m \in D} E_m$, and let F' denote the fibred product $D \times_D \bigcup_{m \in D} E_m = \{(m, n) \,|\, m \in D, n \in E_m\}$. Let $R : F \to F'$ be defined by $R(m, f) = (m, f(m))$ for each $(m, f) \in F$ and let $S : F' \to X$ be a function. If $\lim_m \lim_n S(m, n) \equiv s(\mathcal{C})$, then $(S \circ R, s) \in \mathcal{C}$.

A few comments concerning this definition are in order. First, conditions (a) and (b) reflect elementary properties of net convergence in a topological space. Second, if a net $S\colon D \to X$ does not converge to s in the topological space X, there must exist $U \in N(s)$ and a cofinal subset $D' \subseteq D$ such that $S_n \in X \setminus U$ for all $n \in D'$. This fact is the reason for stipulating condition (c) in the above definition. Third, the iterated limits theorem, see [7, page 69], is the motivation for condition (d) in the definition. Finally, by a product directed set $\prod_{m \in D} I_m$, we understand of course the pointwise ordering on the product $\prod_{m \in D} I_m$ of the directed sets I_m; thus, for elements f and g of $\prod_{m \in D} I_m$, we have $f \leq g$ iff $f(m) \leq g(m)$ for each $m \in D$.

We now record the main theorem concerning convergence classes in net form. This result is basic to the sort of applications we make later in this paper and elsewhere. However, the last part of the proof given in [7, Theorem 9, page 75] appears to be incorrect (the net $\{T \circ U(m, n), n \in E_m\}$ defined there is clearly not defined), and therefore we take the trouble to fill this gap.

2.12 Theorem Let $\mathcal{C}$ be a convergence class in net form for a non-empty set X. For each $A \subseteq X$, let $A^c = \{s \in X \mid \text{there is a net } S \text{ in } A \text{ with } (S, s) \in \mathcal{C}\}$. Then ^{c} is a closure operator on X and hence defines a topology $\mathcal{T}$ on X. Moreover, we have $(S, s) \in \mathcal{C}$ iff $S_n \to s$ with respect to $\mathcal{T}$.

Proof. Following the proof in [7, Theorem 9, pp. 74–75], suppose that it is already established that ^{c} is a closure operator on X, and that convergence $(\mathcal{C})$ of S to s implies convergence $S_n \to s$ relative to $\mathcal{T}$. Then it remains to show that convergence $S_n \to s$ with respect to $\mathcal{T}$ implies that $(S, s) \in \mathcal{C}$. Suppose in fact that $(S, s) \notin \mathcal{C}$. By condition (c) in the definition of $\mathcal{C}$, there exists a subnet $T\colon D \to X$ of S such that for each subnet R of T we have $(R, s) \notin \mathcal{C}$. For each $m \in D$, let $D_m = \{n \in D \mid n \geq m\}$ and let $A_m = T(D_m)$. Since D_m is cofinal in D, we have that $T|_{D_m}$ is a subnet of T which must converge to s with respect to $\mathcal{T}$ since S and, hence T, have this property. Using the elementary facts quoted earlier concerning nets and closure, the fact that ^{c} defines the topology $\mathcal{T}$ and the fact that closure relative to ^{c} is the same thing as closure relative to $\mathcal{T}$, we get $s \in (A_m)^c$ for each $m \in D$. Therefore, we obtain, for each $m \in D$, a net $U(m, \bullet)\colon E_m \to A_m$ with $(U(m, \bullet), s) \in \mathcal{C}$. We apply condition (d) in the definition of $\mathcal{C}$. Let F and R be as defined in condition (d). Then we get $(U \circ R, s) \in \mathcal{C}$. Because we have $U \circ R(m, f) \in A_m$, there exists $n_{m,f} \in D_m$ with $U \circ R(m, f) = T_{n_{m,f}}$ for all $(m, f) \in F$. We define $\varphi : F \to D$ by $\varphi(m, f) = n_{m,f}$ for all $(m, f) \in F$ and obtain $U \circ R = T \circ \varphi$. Finally, given $m \in D$, take any $(m, f) \in F$, that is, choose any f. Then, if $(m', g) \geq (m, f)$, we have $\varphi(m', g) = n_{m',g} \geq m' \geq m$. Therefore, $U \circ R$ is a subnet of T and $(U \circ R, s) \in \mathcal{C}$, which gives a contradiction to our present assumption. We therefore conclude that $(S, s) \in \mathcal{C}$ to finish the proof. ■

We now turn to the main topic of this subsection, namely, the provision of conditions on classes of filters which ensure that they determine a topology in which the resulting convergent filters are precisely the filters first given. The first step is to provide a suitable filter form of the theorem on iterated limits for nets, as follows.

2.13 Theorem Let D be an index set, let $(\mathcal{F}_d)_{d \in D}$ be a family of filters on a topological space $(X, \mathcal{T})$, let $(A_d)_{d \in D}$ be a family of subsets of X such that $\{A_d \mid d \in D\}$ is a filter base on X and let $S = \{s_d \mid d \in D\} \subseteq X, s \in X$. Suppose that for all $d \in D$ we have

$$A_d \in \mathcal{F}_d,\ \mathcal{F}_d \to s_d \text{ and for all } s' \in S \text{ there is } d' \in D \text{ such that } A_d \in \mathcal{F}_{d'},\ \mathcal{F}_{d'} \to s'. \tag{1}$$

Let $\mathcal{F}$ be a filter with $S \in \mathcal{F}$ and such that $\mathcal{F} \to s$. Then there exists a filter $\mathcal{G}$ on X with $A_d \in \mathcal{G}$ for all $d \in D$ and $\mathcal{G} \to s$.

Proof. Suppose that the premises of our claim are satisfied. From condition (1), we conclude $s_d \in S \subseteq \overline{A_d}$ for all $d \in D$. Because $S \in \mathcal{F}$ and $\mathcal{F} \to s$, we obtain $s \in \overline{S}$. In particular, we have $s \in \overline{S} \subseteq \overline{A_d}$ for all $d \in D$. Let $\mathcal{B} = \{A_d \cap U \mid d \in D, U \in N(s)\}$, noting that these sets are non-empty since $s \in \overline{A_d}$ for all d. Then $\mathcal{B}$ is a filter base because, for all $d, d' \in D, U, U' \in N(s)$, there exists $d'' \in D$ with $A_{d''} \subseteq A_d \cap A_{d'}$ and hence

$$\emptyset \neq A_{d''} \cap (U \cap U') \subseteq (A_d \cap A_{d'}) \cap (U \cap U') = (A_d \cap U) \cap (A_{d'} \cap U').$$

Let $\mathcal{G}$ be the filter with base $\mathcal{B}$. We have $N(s) \subseteq \mathcal{G}$ and $A_d \in \mathcal{G}$ for all $d \in D$. In particular, we obtain $\mathcal{G} \to s$. ■

Condition (d) in the definition of convergence classes in filter form given below is inspired by Theorem 2.13. Condition (a) is inspired by considering the filter generated by a constant net, and condition (b) by the fact that every filter $\mathcal{G}$ finer than $\mathcal{F}$ with $\mathcal{F}$ converging to x also converges to x. Finally, condition (c) is clearly necessary by elementary properties of convergence. A similar definition of classes in filter form can be found in [14] where conditions (a) to (c) only of the following definition are used, but with correspondingly weaker conclusions.

2.14 Definition (Convergence Class in Filter Form) Let X be a non-empty set. A *convergence class* $\mathcal{C}$ *for* X *in filter form* is a set of pairs $(\mathcal{F}, s)$ each consisting of a filter $\mathcal{F}$ on X and an element s of X subject to the following conditions. If $(\mathcal{F}, s) \in \mathcal{C}$, we say $\mathcal{F}$ *converges* $(\mathcal{C})$ *to* s and sometimes write $\mathcal{F} \to s\,(\mathcal{C})$.

(a) Let $s \in X$ and let $\mathcal{F} = \{F \subseteq X \,|\, s \in F\} = [s]$ be the point ultrafilter on X at s. Then $(\mathcal{F}, s) \in \mathcal{C}$.

(b) If $(\mathcal{F}, s) \in \mathcal{C}$ and $\mathcal{G}$ is a filter on X finer than $\mathcal{F}$, then $(\mathcal{G}, s) \in \mathcal{C}$.

(c) If $(\mathcal{F}, s) \notin \mathcal{C}$, then there exists a filter $\mathcal{F}' \supseteq \mathcal{F}$ such that, for each filter $\mathcal{G} \supseteq \mathcal{F}'$, we have $(\mathcal{G}, s) \notin \mathcal{C}$.

(d) Let D be an index set, let $(\mathcal{F}_d)_{d\in D}$ be a family of filters on X, let $(A_d)_{d\in D}$ be a family of subsets of X such that $\{A_d \,|\, d \in D\}$ is a filter base on X and let $S = \{s_d \,|\, d \in D\} \subseteq X, s \in X$. Suppose that for all $d \in D$ we have

$$A_d \in \mathcal{F}_d,\ (\mathcal{F}_d, s_d) \in \mathcal{C} \text{ and } \forall\, s' \in S\, \exists\, d' \in D \text{ such that } A_d \in \mathcal{F}_{d'},\ (\mathcal{F}_{d'}, s') \in \mathcal{C}.$$

Let $\mathcal{F}$ be a filter with $S \in \mathcal{F}$ and such that $(\mathcal{F}, s) \in \mathcal{C}$. Then there exists a filter $\mathcal{G}$ on X with $A_d \in \mathcal{G}$ for all $d \in D$ and $(\mathcal{G}, s) \in \mathcal{C}$.

The main theorem concerning convergence classes in filter form is the following analogue of Theorem 2.12.

2.15 Theorem Let $\mathcal{C}$ be a convergence class in filter form for a non-empty set X. For each $A \subseteq X$, let $A^c = \{s \in X \,|\, \text{there is a filter } \mathcal{F} \text{ on } X \text{ with } A \in \mathcal{F} \text{ and } (\mathcal{F}, s) \in \mathcal{C}\}$. Then ^{c} is a closure operator on X and hence defines a topology $\mathcal{T}$ on X. Moreover, we have $(\mathcal{F}, s) \in \mathcal{C}$ iff $\mathcal{F} \to s$ with respect to $\mathcal{T}$.

Proof. First we show that ^{c} is a closure operator on X.
(i) It is clear that $\emptyset^c = \emptyset$.
(ii) We show $A \subseteq A^c$. Let $s \in A$. We have $A \in \mathcal{F} = \{F \subseteq X \,|\, s \in F\} = [s]$. Using condition (a) in the definition of $\mathcal{C}$, we get $(\mathcal{F}, s) \in \mathcal{C}$ and conclude that $s \in A^c$.
(iii) We show $(A \cup B)^c = A^c \cup B^c$. Let $s \in A^c$. Then there exists a filter $\mathcal{F}$ on X with $A \in \mathcal{F}$ and $(\mathcal{F}, s) \in \mathcal{C}$. Since $\mathcal{F}$ is a filter, we obtain that $(A \cup B) \in \mathcal{F}$ and conclude that $s \in (A \cup B)^c$. In the same way, one proves $B^c \subseteq (A \cup B)^c$. Now let $s \in (A \cup B)^c$. Then there exists a filter $\mathcal{F}$ on X with $(A \cup B) \in \mathcal{F}$ and $(\mathcal{F}, s) \in \mathcal{C}$. Let $\mathcal{B}_1 = \{F \cap A \,|\, F \in \mathcal{F}\}$ and $\mathcal{B}_2 = \{F \cap B \,|\, F \in \mathcal{F}\}$. Assume $\emptyset \in \mathcal{B}_1 \cap \mathcal{B}_2$. We obtain $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap A = F_2 \cap B = \emptyset$ and conclude $(F_1 \cap F_2) \cap (A \cup B) = \emptyset$, which contradicts the fact that $(A \cup B), (F_1 \cap F_2) \in \mathcal{F}$. Therefore, $\mathcal{B}_1$ or $\mathcal{B}_2$ is a base for a filter $\mathcal{F}' \supseteq \mathcal{F}$ with $A \in \mathcal{F}'$ or $B \in \mathcal{F}'$. Using condition (b) in the definition of $\mathcal{C}$, we conclude that $(\mathcal{F}', s) \in \mathcal{C}$. So, $s \in A^c \cup B^c$.

(iv) We show $(A^c)^c = A^c$. Using (ii), we have $A^c \subseteq (A^c)^c$. Let $s \in (A^c)^c$. Then there exists a filter $\mathcal{F}$ with $A^c \in \mathcal{F}$ and $(\mathcal{F}, s) \in \mathcal{C}$. For each $a \in A^c$, there exists a filter $\mathcal{F}_a$ with $A \in \mathcal{F}_a$ and $(\mathcal{F}_a, a) \in \mathcal{C}$. We use condition (d) in the definition of $\mathcal{C}$. Let $D = A^c$, let $A_a = A$ and $s_a = a$ for all $a \in D$, and let $S = A^c$. Then the premises of condition (d) are satisfied, and there exists a filter $\mathcal{G}$ on X with $A \in \mathcal{G}$ and $(\mathcal{G}, s) \in \mathcal{C}$. We conclude that $s \in A^c$.

Next we prove the equivalence statement in the theorem.

(v) We show $(\mathcal{F}, s) \in \mathcal{C}$ implies $\mathcal{F} \to s$ with respect to $\mathcal{T}$. Suppose that $\mathcal{F} \to s$ with respect to $\mathcal{T}$ is false. Then there exists $U \in N(s) \cap \mathcal{T}$ with $U \notin \mathcal{F}$. Let $\mathcal{B} = \{F \cap (X \setminus U) \mid F \in \mathcal{F}\}$. Since, for all $F \in \mathcal{F}$, $F \cap (X \setminus U) \neq \emptyset$ (otherwise there would exist $F \in \mathcal{F}$ with $F \subseteq U$, that is, $U \in \mathcal{F}$, which is a contradiction), $\mathcal{B}$ is a base for a filter $\mathcal{F}' \supseteq \mathcal{F}$. Using condition (b) in the definition of $\mathcal{C}$, it follows that $(\mathcal{F}', s) \in \mathcal{C}$. Because $B \subseteq X \setminus U$ for all $B \in \mathcal{B}$, we have $X \setminus U \in \mathcal{F}'$ and so we get $s \in (X \setminus U)^c$ in contradiction to $X \setminus U = (X \setminus U)^c$ and $s \in U$. So, $\mathcal{F} \to s$ with respect to $\mathcal{T}$ is true.

(vi) We show that $\mathcal{F} \to s$ with respect to $\mathcal{T}$ implies that $(\mathcal{F}, s) \in \mathcal{C}$. Suppose $(\mathcal{F}, s) \notin \mathcal{C}$. By condition (c) in the definition of $\mathcal{C}$, there exists a filter $\mathcal{F}' \supseteq \mathcal{F}$ such that, for all filters $\mathcal{G} \supseteq \mathcal{F}'$, we have $(\mathcal{G}, s) \notin \mathcal{C}$. We have $N(s) \subseteq \mathcal{F} \subseteq \mathcal{F}'$. Therefore, we obtain $s \in \bigcap_{F \in \mathcal{F}'} \overline{F}$. The definition of ^{c} and the equality of closure in ^{c} and in $\mathcal{T}$ yields, for each $F \in \mathcal{F}'$, a filter $\mathcal{F}_F$ with $F \in \mathcal{F}_F$ and $(\mathcal{F}_F, s) \in \mathcal{C}$. We use condition (d) in the definition of $\mathcal{C}$. Let $D = \mathcal{F}'$, let $A_F = F$, and let $s_F = s$ for each $F \in \mathcal{F}'$. Due to condition (a) in the definition of $\mathcal{C}$, we get, for the point ultrafilter $\mathcal{F}'' = [s]$, the property $(\mathcal{F}'', s) \in \mathcal{C}$. So, the premises of condition (d) are satisfied and we obtain a filter $\mathcal{G}$ on X with $F \in \mathcal{G}$ for all $F \in \mathcal{F}'$ and $(\mathcal{G}, s) \in \mathcal{C}$. In particular, $\mathcal{G} \supseteq \mathcal{F}'$ which is a contradiction to our present assumption. Thus, we conclude $(\mathcal{F}, s) \in \mathcal{C}$, as required to finish the proof. ■

2.16 Remark It is an immediate consequence of the definitions that each convergence class in net form resp. filter form is a convergence space in net form resp. filter form.

Some natural questions now immediately arise as a consequence of the results above.

(1) Every convergence class induces a topology on the underlying space. This topology induces on the other hand a convergence space (see Lemma 2.5 and Lemma 2.9). Is this convergence space once again a convergence class in net or filter form? If so, is this convergence class equal to the original convergence class?

(2) Is every convergence class in filter form a pointed convergence space? Is the induced topology of the convergence space equal to the induced topology of the convergence class?

(3) Can one transform each convergence class in net form into a convergence class in filter form (and vice versa) such that both induce the same topology?

It is the purpose of the rest of this section to give positive answers to all these questions, and we proceed to do this next. In the following, we denote the induced topology of a convergence space $(X, \mathcal{S})$ or $(X, \mathcal{F})$ by TX. Each convergence class will also be interpreted as a convergence space with extra properties. Therefore, if we speak of a convergence class we will sometimes use the notation employed for convergence spaces to denote elements of the convergence class.

2.4 Properties of Convergence Classes in Net Form

2.17 Lemma Let X be a non-empty set and let $\mathcal{C}$ be a convergence class for X in net form. Then the induced topology $\mathcal{T}$ of the convergence class coincides with TX, that is, $TX = \mathcal{T}$.

Proof. Let $O \in TX$ be an open set of the underlying convergence space in net form. We show that $X \setminus O$ is closed with respect to $\mathcal{T}$. Suppose that there exists $s \in (X \setminus O)^c \cap O$. Then there exists a net $S : D \to X \setminus O$ with $S_n \downarrow s \in O$. Since $O \in TX$, there exists $n \in D$ such that $S_m \in O$ for all $m \geq n$, which contradicts the fact that $S_n \in X \setminus O$ for all $n \in D$. We conclude $(X \setminus O)^c = X \setminus O$ or, in other words, that $O \in \mathcal{T}$.

Now let $O \in \mathcal{T}$ and suppose that $S \downarrow s \in O$, so that $S : D \to X$ is a net and $S_n \to s \in O$ with respect to $\mathcal{T}$. We conclude that there exists $n \in D$ with $S_m \in O$ for all $m \geq n$. Since $s \in O$ was chosen arbitrarily and because of the definition of TX, we get $O \in TX$. ■

2.18 Lemma (Associated Convergence Space) Let X be a non-empty set and let $\mathcal{C}$ be a convergence class for X in net form. Let $\mathcal{T}$ be the induced topology of $\mathcal{C}$, let $(X, \downarrow_{\mathcal{O}})$ be the induced convergence space (see Definition 2.10) with respect to the topology $\mathcal{T}$ and, finally, let $S \downarrow_{\mathcal{C}} s$ iff $(S, s) \in \mathcal{C}$. Then we have $\downarrow_{\mathcal{O}} = \downarrow_{\mathcal{C}}$.

Proof. Applying Definition 2.10 and using Theorem 2.12 we conclude, for each net S in X and $s \in X$, that

$$S \downarrow_{\mathcal{O}} s \iff S_n \to s \text{ with respect to } \mathcal{T} \iff S \downarrow_{\mathcal{C}} s.$$

■

2.19 Lemma Let $(X, \mathcal{T})$ be a topological space. Let $(X, \downarrow_{\mathcal{O}})$ be the induced convergence space in net form with respect to the topology $\mathcal{T}$. We define $(S, s) \in \mathcal{C}$ iff $S \downarrow_{\mathcal{O}} s$. Then $\mathcal{C}$ is a convergence class in net form.

Proof. Since $(X, \downarrow_{\mathcal{O}})$ is a convergence space in net form, conditions (a) and (b) in the definition of convergence class in net form are already satisfied. We have to verify conditions (c) and (d). We have

$$(S, s) \in \mathcal{C} \iff S \downarrow_{\mathcal{O}} s \iff S_n \to s \text{ with respect to } \mathcal{T}.$$

First we prove condition (c). Let $S : D \to X$ be a net and let $(S, s) \notin \mathcal{C}$, so that (S_n) does not converge to s relative to $\mathcal{T}$. Then there is a neighbourhood U of s, and a cofinal subset D' of D such that $S_m \in X \setminus U$ for all $m \in D'$. Then the restriction of S to D' is a subnet T of S with the property that every subnet R of T fails to converge to s relative to $\mathcal{T}$. So, for each subnet R of T, we have $(R, s) \notin \mathcal{C}$, as required.

Condition (d) follows immediately because we defined $(S, s) \in \mathcal{C}$ iff $S_n \to s$ with respect to $\mathcal{T}$. We have only to apply the theorem on iterated limits ([7, Page 69]) to finish the argument. ■

2.5 Properties of Convergence Classes in Filter Form

2.20 Lemma Let X be a non-empty set and let $\mathcal{C}$ be a convergence class for X in filter form. Then the induced topology $\mathcal{T}$ of the convergence class coincides with TX, that is, $TX = \mathcal{T}$.

Proof. Let $O \in TX$ be an open set of the underlying convergence space in filter form. We show that $X \setminus O$ is closed with respect to $\mathcal{T}$. Assume that there exists $s \in (X \setminus O)^c \cap O$. Then there exists a filter $\mathcal{F}$ on X with $X \setminus O \in \mathcal{F}$ and $\mathcal{F} \downarrow s \in O$. Since $O \in TX$, we conclude that $O \in \mathcal{F}$, which contradicts $X \setminus O \in \mathcal{F}$. Thus, we get $X \setminus O = (X \setminus O)^c$, so that O is open with respect to $\mathcal{T}$.

Now let $O \in \mathcal{T}, O \neq \emptyset$. Let $\mathcal{A}$ be a filter on X with $\mathcal{A} \downarrow x \in O$. Assume that $O \notin \mathcal{A}$. Then, for all $B \subseteq O$, we have $B \notin \mathcal{A}$, and we conclude that for all $B \in \mathcal{A}$ we have $B \cap (X \setminus O) \neq \emptyset$. We define $\mathcal{B} = \{B \cap (X \setminus O) \mid B \in \mathcal{A}\}$. Then $\mathcal{B}$ is a base for a filter $\mathcal{G} \supseteq \mathcal{A}$. Therefore, we obtain $\mathcal{G} \downarrow x \in O$ and $X \setminus O \in \mathcal{G}$. This means that $x \in (X \setminus O)^c = X \setminus O$ as $O \in \mathcal{T}$, and we get a contradiction. Thus, $O \in \mathcal{A}$ and, as $x \in O$ is arbitrary, we conclude that $O \in TX$. ■

2.21 Lemma (Associated Convergence Space) Let X be a non-empty set and let $\mathcal{C}$ be a convergence class for X in filter form. Let $\mathcal{T}$ be the induced topology of $\mathcal{C}$, let $(X, \downarrow_{\mathcal{O}})$ be the induced convergence space (see Definition 2.6) with respect to the topology $\mathcal{T}$ and, finally, let $\mathcal{F} \downarrow_{\mathcal{C}} x$ iff $(\mathcal{F}, x) \in \mathcal{C}$. Then we have $\downarrow_{\mathcal{O}} = \downarrow_{\mathcal{C}}$.

Proof. Applying Definition 2.6 and using Theorem 2.15 we conclude, for each filter $\mathcal{F}$ on X and $x \in X$, that

$$\mathcal{F} \downarrow_{\mathcal{O}} x \iff \mathcal{F} \to x \text{ with respect to } \mathcal{T} \iff \mathcal{F} \downarrow_{\mathcal{C}} x.$$

■

2.22 Corollary Let X be a non-empty set. Then every convergence class $\mathcal{C}$ for X in filter form is a pointed convergence space in filter form.

Proof. Let $x \in X$. We have $\bigcap \mathcal{F}_x = \bigcap\{\mathcal{A} \mid \mathcal{A} \downarrow_{\mathcal{C}} x\} = \bigcap\{\mathcal{A} \mid \mathcal{A} \downarrow_{\mathcal{O}} x\}$, and the definition of an induced convergence space in filter form (Definition 2.6) yields $\bigcap \mathcal{F}_x = N(x) \in \mathcal{F}_x$. ■

2.23 Lemma Let $(X, \mathcal{T})$ be a topological space. Let $(X, \downarrow_{\mathcal{O}})$ be the induced convergence space in filter form with respect to the topology $\mathcal{T}$. We define $(\mathcal{F}, x) \in \mathcal{C}$ iff $\mathcal{F} \downarrow_{\mathcal{O}} x$. Then $\mathcal{C}$ is a convergence class in filter form.

Proof. Since $(X, \downarrow_{\mathcal{O}})$ is a convergence space in filter form, conditions (a) and (b) in the definition of convergence class in filter form are already satisfied. We have to verify conditions (c) and (d). We have

$$(\mathcal{F}, x) \in \mathcal{C} \iff \mathcal{F} \downarrow_{\mathcal{O}} x \iff \mathcal{F} \supseteq N(x) \iff \mathcal{F} \to x \text{ with respect to } \mathcal{T}.$$

First we prove condition (c). Let $(\mathcal{F}, s) \notin \mathcal{C}$. There exists $U \in N(s)$ with $U \notin \mathcal{F}$. Since $\mathcal{F}$ is a filter, for all $B \subseteq U$ we have $B \notin \mathcal{F}$ and we conclude that, for all $F \in \mathcal{F}$, we have $F \cap (X \setminus U) \neq \emptyset$. Let $\mathcal{B} = \{F \cap (X \setminus U) \mid F \in \mathcal{F}\}$. Then $\mathcal{B}$ is a base for a filter $\mathcal{F}' \supseteq \mathcal{F}$. Now let $\mathcal{G} \supseteq \mathcal{F}'$ be a filter on X. Assume that $U \in \mathcal{G}$. Because $\mathcal{B} \subseteq \mathcal{G}$, there exists $F \in \mathcal{F}$ with $F \cap (X \setminus U) \in \mathcal{B} \subseteq \mathcal{F}' \subseteq \mathcal{G}$ and $\emptyset = U \cap F \cap (X \setminus U) \in \mathcal{G}$, which is a contradiction. Thus, $U \notin \mathcal{G}$ and, for all filters $\mathcal{G} \supseteq \mathcal{F}'$, we get $(\mathcal{G}, s) \notin \mathcal{C}$, so that condition (c) is satisfied.

Condition (d) follows immediately because we defined $(\mathcal{F}, s) \in \mathcal{C}$ iff $\mathcal{F} \to s$ with respect to $\mathcal{T}$. We have only to apply Theorem 2.13 to complete the proof. ■

2.24 Remark Lemma 2.18 and Lemma 2.19 resp. Lemma 2.21 and Lemma 2.23 imply that, for every convergence class $\mathcal{C}$ for X in net form resp. filter form, there exists a topology $\mathcal{T}$ on X such that the induced convergence space is the convergence class $\mathcal{C}$, and that every topology $\mathcal{T}$ on X induces a topological convergence space in net form resp. filter form which is a convergence class in net form resp. filter form. Thus, the class of topological convergence spaces in net form resp. filter form is exactly the class of all convergence classes in net form resp. filter form. Indeed, the two conditions (c) and (d), in either definition of a convergence class, give the conditions needed for the convergence structure provided by a convergence space (in net form) (resp. in filter form) to be topological.

2.6 Interchange of Form in Convergence Classes

In the following, we will denote the closure operator of a convergence class in net form by ^{c_1}, and the closure operator of a convergence class in filter form by ^{c_2}. Let X be a non-empty set and let $\mathcal{C}$ be a convergence class on X in net form. By Theorem 2.12, ^{c_1} defines a topology $\mathcal{T}$ on X with the property $(S,s) \in \mathcal{C}$ iff $S_n \to s$ with respect to $\mathcal{T}$. As we have just seen in Lemma 2.23, $\mathcal{T}$ defines a convergence class $\mathcal{C}'$ on X in filter form with the property

$$(\mathcal{F}, s) \in \mathcal{C}' \iff \mathcal{F} \downarrow_{\mathcal{O}} s \iff \mathcal{F} \supseteq N(s) \iff \mathcal{F} \to s \text{ with respect to } \mathcal{T}.$$

Thus, the induced topologies of $\mathcal{C}$ and $\mathcal{C}'$ are equal, that is, we have $A^{c_1} = A^{c_2}$ for all $A \subseteq X$. In other words, we have for all $A \subseteq X$

$$\begin{aligned} A^{c_1} &= \{s \in X \,|\, \text{there exists net } S \text{ in } A \text{ such that } (S,s) \in \mathcal{C}\} \\ &= \{s \in X \,|\, \text{there exists filter } \mathcal{F} \text{ such that } A \in \mathcal{F} \text{ and } (\mathcal{F}, s) \in \mathcal{C}'\} = A^{c_2}. \end{aligned}$$

In fact, we can construct $\mathcal{C}'$ from $\mathcal{C}$ by means of the following result.

2.25 Lemma Let $X, \mathcal{C}$ and $\mathcal{C}'$ be as defined above. Then we have that $(\mathcal{F}, s) \in \mathcal{C}'$ iff there exists a net $S : D \to X$ such that $(S,s) \in \mathcal{C}$ and $\mathcal{B} = \{B_n \,|\, n \in D\}$ is a base for $\mathcal{F}$, where $B_n = \{S_m \,|\, m \geq n\}$.

Proof. For sufficiency, let $(S,s) \in \mathcal{C}$ and suppose that $O \in N(s)$ is open. Then there exists $n \in D$ such that $S_m \in O$ for all $m \geq n$, and we have $B_n \subseteq O$. Since $\mathcal{F}$ is a filter, we get $O \in \mathcal{F}$ and therefore we have $N(s) \subseteq \mathcal{F}$. Thus, we conclude $(\mathcal{F}, s) \in \mathcal{C}'$ by the definition of $\mathcal{C}'$.

Conversely, let $(\mathcal{F}, s) \in \mathcal{C}'$. By definition, $\mathcal{F} \to s$ with respect to $\mathcal{T}$. Let $S : D_{\mathcal{F}} \to X$ be the net based on $\mathcal{F}$. By Theorem 2.1 (a), it follows that $S_n \to s$ with respect to $\mathcal{T}$. Thus, we obtain $(S,s) \in \mathcal{C}$ by means of the definition of $\mathcal{C}$. The construction of S yields $B_{(x,F)} = \{S_{(x',F')} \,|\, (x',F') \geq (x,F)\} = F$ for all $x \in F \in \mathcal{F}$. In particular, we have $\mathcal{B} = \{B_{(x,F)} \,|\, (x,F) \in D_{\mathcal{F}}\} = \mathcal{F}$ is a base for $\mathcal{F}$. ■

Next, working in the opposite direction, we start with a convergence class $\mathcal{C}'$ for X in filter form. Let $\mathcal{T}$ be the induced topology, so that we have

$$(\mathcal{F}, s) \in \mathcal{C}' \iff \mathcal{F} \to s \ \text{ with respect to } \mathcal{T} \iff \mathcal{F} \supseteq N(s).$$

As we have seen in Lemma 2.19, $\mathcal{T}$ defines a convergence class $\mathcal{C}$ for X in net form and, by means of Lemma 2.17, the induced topologies of $\mathcal{C}$ and $\mathcal{C}'$ coincide and, hence, are equal to $\mathcal{T}$. Furthermore, $\mathcal{C}$ has the property

$$(S,s) \in \mathcal{C} \iff \text{there is a net } S : D \to X \text{ in } X \text{ such that } S_n \to s \text{ with respect to } \mathcal{T}.$$

Once again Lemma 2.25 holds. In addition, we have the following result.

2.26 Lemma Let $X, \mathcal{C}, \mathcal{C}'$ and $\mathcal{T}$ be defined as above. Then we have $(S, s) \in \mathcal{C}$ iff there exists $(\mathcal{F}, s) \in \mathcal{C}'$ such that S is a subnet of T, where $D_{\mathcal{F}} = \{(x, F) \,|\, x \in F \in \mathcal{F}\}$ and T is the net $T : D_{\mathcal{F}} \to X$ defined by $T(x, F) = x$.

Proof. For the necessity, let $(S, s) \in \mathcal{C}$, so that $S : D \to X$ is a net with $S_n \to s$ in the topology $\mathcal{T}$. Let $B_n = \{S_m \,|\, m \geq n\}$, and let $\mathcal{B} = \{B_n \,|\, n \in D\}$ be the base for the filter $\mathcal{F}$ generated by the net S. Using Theorem 2.1 (b), we see that $\mathcal{F} \to s$ with respect to $\mathcal{T}$. Thus, we conclude that $(\mathcal{F}, s) \in \mathcal{C}'$. Let $D_{\mathcal{F}}$ and T be as defined in our claim, and let $\varphi : D \to D_{\mathcal{F}}$ be defined by $\varphi(n) = (S_n, B_n)$ for all $n \in D$. Then, for all $n, m \in D$, we have that $n \leq m$ implies $B_m \subseteq B_n$ which in turn implies that $(S_n, B_n) \leq (S_m, B_m)$. So, φ is a monotonic mapping. Since, for all $x \in F \in \mathcal{F}$, there exists $n \in D$ such that $F \supseteq B_n$, we obtain $(x, F) \leq (S_n, B_n)$ so that the image of φ is cofinal in $D_{\mathcal{F}}$. Therefore, $T \circ \varphi$ is a net with $T(\varphi(n)) = T((S_n, B_n)) = S_n = S(n)$, that is, $S = T \circ \varphi$. Thus, S is a subnet of T, as required.

Conversely, let $(\mathcal{F}, s) \in \mathcal{C}'$. By definition of $\mathcal{C}'$, we have $\mathcal{F} \to s$ with respect to $\mathcal{T}$. Because T is the net based on $\mathcal{F}$, using Theorem 2.1 (a) we conclude that $T_n \to s$ with respect to $\mathcal{T}$. Since S is a subnet of T, we obtain also that $S_n \to s$ with respect to $\mathcal{T}$ so that $(S, s) \in \mathcal{C}$ by definition of $\mathcal{C}$. ■

3 Convergence Classes and VDM

As mentioned in the introduction, the paper falls naturally into two parts. The first of these is the previous section in which we established a rather satisfactory theory of convergence. This section constitutes the second part of the paper, and in it we want to apply certain of the convergence results of the first part to spaces of partial functions and to certain natural operators that they carry.

We begin by establishing some preliminaries and some notation.

3.1 Preliminaries

By the term *monoid*, we mean a non-empty set M endowed with a closed and associative binary operation $*$, called the law of composition or multiplication, which possesses an identity element u for the composition. There are several monoids of interest here, and two such examples of particular importance are $(\mathcal{P}(X), \cup, \emptyset)$ and $(\mathcal{P}(X), \cap, X)$, where $\mathcal{P}(X)$ again denotes the power set of X. In the first, the law of composition is the union of sets and the identity is the empty set; in the second the law of composition is the intersection of sets and the identity is the whole set X. We say that a monoid $(M, *, u)$ is a *topological monoid* if M is a topological space and the law of composition $*$ is a continuous function on $M \times M$, where $M \times M$ is endowed with the usual product topology determined by the topology on M.

We shall use the term *domain* (or *Scott domain*) with the meaning employed in [15], which is our general reference to this subject. Thus, a *domain* $(D, \sqsubseteq, \perp)$, or simply D when no confusion is caused, is a consistently complete algebraic complete partial order. We let D_c denote the set of compact elements of D, and, given $x \in D$, we let $\mathrm{approx}(x)$ denote the set $\{a \in D_c; a \sqsubseteq x\}$. Of course, $\mathrm{approx}(x)$ is directed and $x = \sup \mathrm{approx}(x)$ for each $x \in D$, where in general $\sup A$ denotes the supremum of the directed set A. Any complete partial order (cpo), and hence any domain, may be endowed with the well-known Scott topology, see

[2, 15], in which a set O is open if and only if it satisfies: (i) whenever $x \in O$ and $x \sqsubseteq y$, then $y \in O$, and (ii) whenever A is directed and $\sup A \in O$, then $A \cap O \neq \emptyset$. In the case of a domain, this topology has a rather simple description in that the collection $\{\uparrow a; a \in D_c\}$ is a basis for the Scott topology, where $\uparrow x = \{y \in D; x \sqsubseteq y\}$ for any $x \in D$.

One point of notation to which we should draw the attention of the reader is the following. In Section 2, we followed [7] closely and therefore we used the symbol D for the index set of nets. From now on, since we are following [15] closely, D will usually denote a domain, and therefore we will use I or J etc. to indicate index sets for nets and directed sets in general, and $i, j, n, m, \alpha, \beta$ etc. to denote elements of these index sets.

It will be useful to record next a couple of elementary facts we will use without further mention. The first is the well-known formulation of Scott continuity of functions between domains in terms of order properties, see [15, Proposition 5.2.3]. *Let D and E be domains. Then a function $f : D \to E$ is continuous with respect to the Scott topology if and only if it satisfies the property: whenever $A \subseteq D$ is directed, we have that $f[A]$ is directed in E and* $f(\sup A) = \sup f[A]$. And the second concerns cartesian products, see [15, Proposition 2.2.4]. *Let D, E and F be complete partial orders. Then a function $f : D \times E \to F$ is continuous if and only if f is continuous in each argument.*

Of course, the power set $\mathcal{P}(X)$ of a non-empty set X is a domain, ordered by set inclusion, whose compact elements are the finite sets. It is sometimes useful to identify $\mathcal{P}(X)$ with the set of all total functions from X to $\mathbf{2}$, by means of the characteristic functions of subsets of X, or with the product $\Pi_{i \in X}\, \mathbf{2}_i$ of X copies of $\mathbf{2}$, where $\mathbf{2}$ denotes the two-element set $\{0, 1\}$. The usual product topology on $\Pi_{i \in X}\, \mathbf{2}_i$, when $\mathbf{2}$ is endowed with the Scott topology, results in the Scott topology on $\Pi_{i \in X}\, \mathbf{2}_i$ and hence on $\mathcal{P}(X)$. Alternatively, we may endow $\mathbf{2}$ with the discrete topology. Then, we will call the resulting topology on $\mathcal{P}(X)$, a *Cantor topology* since $\Pi_{i \in X}\, \mathbf{2}_i$ is homeomorphic to the Cantor set in the real line whenever X is denumerable. This topology also has significance in computing because of its well-known role in domain theory in relation to sets of maximal elements and universal domains. Moreover, it coincides with the Lawson topology on $\mathcal{P}(X)$ (the Lawson topology is the common refinement of the Scott topology and the lower topology, see [2]). Finally, it has an important role in logic programming semantics (see [12]) and in termination of logic programs, see [4]. In the present work, it turns out to be important in handling the override operator.

We have a simple characterization of net convergence in the Scott topology.

3.1 Proposition Let D be a domain. A net $x_i \to x$ in the Scott topology on D if and only if for each $a \in \mathrm{approx}(x)$ there is an index i_0 such that $a \sqsubseteq x_i$ whenever $i_0 \leq i$.

Proof. Suppose that $x_i \to x$ and that $a \in \mathrm{approx}(x)$. Then $\uparrow a$ is a Scott neighbourhood of x, and x_i eventually belongs to $\uparrow a$. So certainly there is an i_0 with the stated property.

Conversely, suppose that a net (x_i) and an element x are given in D and that the stated condition on elements $a \in \mathrm{approx}(x)$ holds. Given an arbitrary Scott neighbourhood U of x, there is a basic neighbourhood $\uparrow a$ of x inside U, where $a \in \mathrm{approx}(x)$. But the stated condition now simply asserts that there is an i_0 such that $x_i \in U$ whenever $i_0 \leq i$ and therefore $x_i \to x$, as required. ∎

The following are simple, but useful, technical facts concerning the two topologies we have been discussing on $\mathcal{P}(X)$; the first follows from Proposition 3.1.

3.2 Proposition (1) In the Scott topology on $\mathcal{P}(X)$, a net A_i of sets converges to a set A if and only if every element of A is eventually an element of A_i.
(2) In the Cantor topology on $\mathcal{P}(X)$, a net A_i converges to A iff every element of A is eventually an element of A_i, and every element of X not in A is eventually not in A_i.

3.3 Example It is worth noting that, in each case, the stated conditions in Proposition 3.1 and in Proposition 3.2 actually define convergence classes generating the corresponding topology. Thus, they provide examples of topologies of interest in computing given in these terms.

3.4 Proposition In either the Scott topology or the Cantor topology on $\mathcal{P}(X)$, both $(\mathcal{P}(X), \cup, \emptyset)$ and $(\mathcal{P}(X), \cap, X)$ are topological monoids.

Proof. We show that $(\mathcal{P}(X), \cup, \emptyset)$ is a topological monoid in the Cantor topology, the arguments for the other claims being similar.

Suppose that $(A, B)_i = (A_i, B_i)$ is a net in $\mathcal{P}(X) \times \mathcal{P}(X)$ converging to (A, B) in the product of the Cantor topologies on $\mathcal{P}(X)$, thus A_i converges to A and B_i converges to B in $\mathcal{P}(X)$. Suppose $x \in A \cup B$. If $x \in A$, then, by Proposition 3.2, x is eventually in A_i and hence x is eventually in $A_i \cup B_i$, and similarly if it is the case that $x \in B$. If $x \notin A \cup B$, then x is not in A and x is not in B. Hence, by Proposition 3.2 again, x is eventually not in A_i and is eventually not in B_i, and hence x is eventually not in $A_i \cup B_i$. Thus, $A_i \cup B_i$ converges to $A \cup B$, and so $\cup$ is continuous, as required. ■

The following result is proved similarly, and we omit the proof.

3.5 Proposition The mapping $\mathrm{comp} : \mathcal{P}(X) \to \mathcal{P}(X)$ determined by taking the complement of a set and defined by $\mathrm{comp}(S) = X \setminus S$ is continuous in the Cantor topology. Indeed, in the Cantor topology this mapping is a homeomorphism of $\mathcal{P}(X)$ onto itself and is an isomorphism between the topological monoids $(\mathcal{P}(X), \cup, \emptyset)$ and $(\mathcal{P}(X), \cap, X)$.

Since comp is not even monotonic, it is clearly not Scott continuous on $(\mathcal{P}(X), \subseteq)$. It results from this that the override operator is not Scott continuous.

3.6 Definition Let M be a topological monoid and let D be a domain which is also a topological space. Then M will be said to act on (the left of) D if there is a continuous function $\odot : M \times D \to D$, usually written $(m, x) \mapsto m \odot x$, with the following properties:
(i) $u \odot x = x$ for all $x \in D$.
(ii) $m_1 \odot (m_2 \odot x) = (m_1 * m_2) \odot x$ for all $m_1, m_2 \in M$ and all $x \in D$.
(iii) $m \odot a \in D_c$ for all $m \in M$ and all $a \in D_c$.

Given an action of M on D, fixing $m \in M$ determines a continuous function $x \mapsto m \odot x$ of D to itself which preserves the compact elements. Similarly, fixing $x \in D$ determines a continuous function $m \mapsto m \odot x$ from M to D.

3.2 The Basic Operators in VDM and in VDM♣

Let X and Y be sets, and let $(X \to Y)$ denote the set of partial functions mapping X to Y. It is well-known that $(X \to Y)$ is a domain when ordered by graph inclusion: $\mu \sqsubseteq \nu$ if and only if $\mathrm{graph}(\mu) \subseteq \mathrm{graph}(\nu)$, where $\mathrm{graph}(\mu) = \{(x, y) \in X \times Y; x \in \mathrm{dom}(\mu) \text{ and } y = \mu(x)\}$, and here and elsewhere $\mathrm{dom}(\mu)$ denotes the domain of μ. Moreover, if $A = \{\mu_\alpha; \alpha \in I\}$ is a directed set of elements of $(X \to Y)$, then the supremum of A is the partial function well-defined by the union of the graphs of the $\mu_\alpha, \alpha \in I$. Finally, the compact elements of $(X \to Y)$ are the partial functions μ for which graph(μ) is a finite set. We shall always suppose that $(X \to Y)$ is ordered in the way just described. However, we will need to endow $(X \to Y)$ with topologies other than the Scott topology, as well as with the Scott topology, in what follows. At any given time, unless stated otherwise, subsets of $(X \to Y) \times (X \to Y)$ will be given the subspace topology of the product with itself of whatever topology we are considering at that time on $(X \to Y)$.

For convenience, we state next a simple criterion for convergence in the Scott topology on $(X \to Y)$ which follows immediately from Proposition 3.1.

3.7 Proposition A net μ_i converges to μ in the Scott topology on $(X \to Y)$ iff whenever $(x, y) \in \mathrm{graph}(\mu)$ we have $(x, y) \in \mathrm{graph}(\mu_i)$ eventually.

The operators which occur in VDM$^\clubsuit$ are operators defined on $(X \to Y)$. As already noted, it is our aim to study them from the domain-theoretic point of view and to determine the extent to which they are Scott continuous or otherwise. In fact, we work rather more generally than this since we formulate the results in terms of (continuous) actions of monoids on $(X \to Y)$, and obtain the results relative to the usual operators in VDM$^\clubsuit$ by fixing one or other of the arguments. It will be convenient to break the discussion into two parts, namely, into those which are Scott continuous, and those which are not. For general references to the details of the operator calculus used in VDM$^\clubsuit$, we cite [9, 6]. In fact, the basic operators we study here are common to both VDM and VDM$^\clubsuit$.

3.3 Scott-Continuous Operators

In this subsection, the term "continuous" will mean Scott continuous unless otherwise stated.

3.3.1 The Extension Operator, $\sqcup$

Let μ and ν be elements of $(X \to Y)$ which satisfy $\mathrm{dom}(\mu) \cap \mathrm{dom}(\nu) = \emptyset$. We define the *extension* $\mu \sqcup \nu \in (X \to Y)$ *of* μ *by* ν as follows:

$$(\mu \sqcup \nu)(x) = \begin{cases} \mu(x) & \text{if } \ x \in \mathrm{dom}(\mu), \\ \nu(x) & \text{if } \ x \in \mathrm{dom}(\nu). \end{cases}$$

3.8 Theorem The mapping $(\mu, \nu) \mapsto \mu \sqcup \nu$ is Scott continuous as a mapping on the set $(X \to Y) \times_\sqcup (X \to Y)$ of those pairs (μ, ν) in $(X \to Y) \times (X \to Y)$ which satisfy $\mathrm{dom}(\mu) \cap \mathrm{dom}(\nu) = \emptyset$.

Proof. We begin by showing that $(X \to Y) \times_\sqcup (X \to Y)$ is a subdomain of $(X \to Y) \times (X \to Y)$. Let $A = \{(\mu_\alpha, \nu_\alpha); \alpha \in I\}$ be a directed set in $(X \to Y) \times_\sqcup (X \to Y)$. Then A is directed as a subset of $(X \to Y) \times (X \to Y)$. Hence, $\pi_0[A]$ and $\pi_1[A]$ are directed sets in $(X \to Y)$, where π_0 and π_1 are the projections on the first and second factors respectively. Moreover, the supremum of A in $(X \to Y) \times (X \to Y)$ is the pair $(\sup \pi_0[A], \sup \pi_1[A])$, see [15, Lemma 2.2.2]. We show that this pair belongs to $(X \to Y) \times_\sqcup (X \to Y)$ and hence that it is the supremum of A in $(X \to Y) \times_\sqcup (X \to Y)$. Suppose that $\mathrm{dom}(\sup \pi_0[A]) \cap \mathrm{dom}(\sup \pi_1[A]) \neq \emptyset$, and that x is an element of this non-empty set. Then there are α and β in I such that $(\sup \pi_0[A])(x) = \mu_\alpha(x)$ and $(\sup \pi_1[A])(x) = \nu_\beta(x)$. Since A is directed, there is $\gamma \in I$ with $(\mu_\alpha, \nu_\alpha) \sqsubseteq (\mu_\gamma, \nu_\gamma)$

and $(\mu_\beta, \nu_\beta) \sqsubseteq (\mu_\gamma, \nu_\gamma)$. But this leads to the conclusion that $(\sup \pi_0[A])(x) = \mu_\gamma(x)$ and $(\sup \pi_1[A])(x) = \nu_\gamma(x)$ and hence to the contradiction that $\mathrm{dom}(\mu_\gamma) \cap \mathrm{dom}(\nu_\gamma) \neq \emptyset$. Thus, it now follows that $(X \to Y) \times_\sqcup (X \to Y)$ is a subcpo of $(X \to Y) \times (X \to Y)$ and indeed it is readily checked that it is in fact a subdomain.

For the stated continuity, it suffices, by symmetry, to check continuity in either argument, so fix μ and consider the map $\nu \mapsto \mu \sqcup \nu$. If (μ, ν_1) and (μ, ν_2) are elements of $(X \to Y) \times_\sqcup (X \to Y)$ and $\nu_1 \sqsubseteq \nu_2$, then it is easily seen that $\mu \sqcup \nu_1 \sqsubseteq \mu \sqcup \nu_2$ so that the map in question is monotonic.

Now suppose that $A = \{\nu_\alpha; \alpha \in I\}$ is a directed set in $(X \to Y)$, where $\mathrm{dom}(\mu) \cap \mathrm{dom}(\nu_\alpha) = \emptyset$ for all $\alpha \in I$, and let $\nu = \sup A$. By the first part of the proof, we know that $\mathrm{dom}(\mu) \cap \mathrm{dom}(\nu) = \emptyset$. If $x \in \mathrm{dom}(\mu)$, then clearly $(\mu \sqcup \sup_{\alpha \in I} \nu_\alpha)(x) = (\mu \sqcup \nu)(x) = \mu(x)$. On the other hand, $(\mu \sqcup \nu_\alpha)(x) = \mu(x)$ for all α and hence $(\sup_{\alpha \in I}(\mu \sqcup \nu_\alpha))(x) = \mu(x)$. If $x \in \mathrm{dom}(\nu)$, then $\nu(x) = \nu_\beta(x)$ for some $\beta \in I$, and so $(\mu \sqcup \nu)(x) = \nu(x) = \nu_\beta(x) = (\mu \sqcup \nu_\beta)(x) = (\sup_{\alpha \in I}(\mu \sqcup \nu_\alpha))(x)$. Thus, $\mu \sqcup \sup_{\alpha \in I} \nu_\alpha = \sup_{\alpha \in I}(\mu \sqcup \nu_\alpha)$, and we have the required continuity. ■

3.3.2 The Glueing Operator, $\cup$

Let μ and ν be elements of $(X \to Y)$ which coincide on the intersection of their domains. Then μ may be *glued* to ν to obtain the partial map $\mu \cup \nu \in (X \to Y)$ defined as follows:

$$(\mu \cup \nu)(x) = \begin{cases} \mu(x) & \text{if } \ x \in \mathrm{dom}(\mu), \\ \nu(x) & \text{if } \ x \in \mathrm{dom}(\nu). \end{cases}$$

3.9 Theorem The mapping $(\mu, \nu) \mapsto \mu \cup \nu$ is Scott continuous as a mapping on the set $(X \to Y) \times_\cup (X \to Y)$ of those pairs (μ, ν) in $(X \to Y) \times (X \to Y)$ which coincide on the intersection of their domains.

Proof. The proof of this result is similar to the proof of the previous result, and we omit the details. ■

3.3.3 The Domain Restriction Operator, $\lhd$

Given $\mu \in (X \to Y)$ and an element S of $\mathcal{P}(X)$, we define the *restriction of μ by S* to be the partial function in $(X \to Y)$, denoted by $\lhd_S \mu$, which satisfies: (i) $\mathrm{dom}(\lhd_S \mu) = S \cap \mathrm{dom}(\mu)$, and (ii) $\lhd_S \mu$ coincides with μ on $S \cap \mathrm{dom}(\mu)$.

3.10 Theorem Suppose that $(X \to Y)$ is endowed with the Scott topology and that $\mathcal{P}(X)$ is endowed with either (1) the Scott topology, or (2) the Cantor topology. Then in either case, the mapping $\lhd : \mathcal{P}(X) \times (X \to Y) \to (X \to Y)$ defined by $\lhd(S, \mu) = \lhd_S \mu$ determines an action of the topological monoid $(\mathcal{P}(X), \cap, X)$ on the domain $(X \to Y)$.

Proof. That $(\mathcal{P}(X), \cap, X)$ is a topological monoid in either case was shown in Proposition 3.4. If S_1 and S_2 are elements of $\mathcal{P}(X)$ and $\mu \in (X \to Y)$, then $S_1 \cap (S_2 \cap \mathrm{dom}(\mu)) = (S_1 \cap S_2) \cap \mathrm{dom}(\mu)$, and it follows that (ii) of Definition 3.6 is satisfied. The other two statements in this definition are clear, and so the result will follow as soon as we have established the required continuity.

(1) For this case, it suffices to check continuity in each argument separately.

Fix μ and consider the mapping $\theta : \mathcal{P}(X) \to (X \to Y)$ defined by $\theta(S) = \lhd_S \mu$. If $S_1 \subseteq S_2$, then $\mathrm{graph}(\lhd_{S_1} \mu) \subseteq \mathrm{graph}(\lhd_{S_2} \mu)$, and so θ is monotonic. Suppose that $A = \{S_\alpha; \alpha \in I\}$ is a directed family of sets in $\mathcal{P}(X)$ and let $S = \sup A = \bigcup_{\alpha \in I} S_\alpha$. We want to establish that $\lhd_{\bigcup_{\alpha\in I} S_\alpha} \mu = \sup_{\alpha \in I}(\lhd_{S_\alpha} \mu)$, for which it suffices to show that $\mathrm{graph}(\lhd_{\bigcup_{\alpha\in I} S_\alpha} \mu) = \bigcup_{\alpha\in I} \mathrm{graph}(\lhd_{S_\alpha} \mu)$ and this is a straightforward calculation.

Now fix S and consider the mapping $\phi : (X \to Y) \to (X \to Y)$ defined by $\phi(\mu) = \lhd_S \mu$. If $\mu_1 \sqsubseteq \mu_2$, that is, $\mathrm{graph}(\mu_1) \subseteq \mathrm{graph}(\mu_2)$, then it is clear that $\mathrm{graph}(\lhd_S \mu_1) \subseteq \mathrm{graph}(\lhd_S \mu_2)$ so that ϕ is monotonic. Suppose that $A = \{\mu_\alpha; \alpha \in I\}$ is a directed set in $(X \to Y)$, and let $\mu = \sup A$. We want to show that $\lhd_S \mu = \sup_{\alpha\in I}(\lhd_S \mu_\alpha)$, and again it is a straightforward calculation to show that $\mathrm{graph}(\lhd_S \mu) = \bigcup_{\alpha\in I} \mathrm{graph}(\lhd_S \mu_\alpha)$, which suffices.

(2) Suppose that the net $(S_i, \mu_i) \to (S, \mu)$ in the product space $\mathcal{P}(X) \times (X \to Y)$, where $\mathcal{P}(X)$ carries the Cantor topology. Then we have $S_i \to S$ in $\mathcal{P}(X)$ and $\mu_i \to \mu$ in $(X \to Y)$. We want to show that $\lhd_{S_i} \mu_i \to \lhd_S \mu$ in the Scott topology on $(X \to Y)$. Suppose that $\nu \in \mathrm{approx}(\lhd_S \mu)$ is arbitrary. By Proposition 3.1, it suffices to show that there is an index i_0 such that $\nu \sqsubseteq \lhd_{S_i} \mu_i$ whenever $i_0 \le i$ or, equivalently, that $\mathrm{graph}(\nu) \subseteq \mathrm{graph}(\lhd_{S_i} \mu_i)$ whenever $i_0 \le i$. Since $\nu \in \mathrm{approx}(\lhd_S \mu)$, we have $\mathrm{graph}(\nu) \subseteq \mathrm{graph}(\lhd_S \mu)$ and hence $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(\lhd_S \mu)$ from which we obtain that $\mathrm{dom}(\nu) \subseteq S$. But ν is a finite function, so that $\mathrm{dom}(\nu)$ is a finite set, and $S_i \to S$ in the Cantor topology. Therefore, by applying Proposition 3.2 (2) as many times as there are elements in $\mathrm{dom}(\nu)$, we see that there exists an index i_1 such that $\mathrm{dom}(\nu) \subseteq S_i$ whenever $i_1 \le i$.

Next, we note that $\nu \sqsubseteq \lhd_S \mu \sqsubseteq \mu$ and so $\nu \in \mathrm{approx}(\mu)$. Since $\mu_i \to \mu$ in the Scott topology, by Proposition 3.1 there is an index i_2 such that $\nu \sqsubseteq \mu_i$ whenever $i_2 \le i$. Choose an index i_0 such that $i_1 \le i_0$ and $i_2 \le i_0$. Then, whenever $i_0 \le i$, we have $\mathrm{graph}(\nu) \subseteq \mathrm{graph}(\mu_i)$ and hence $\mathrm{dom}(\nu) \subseteq \mathrm{dom}(\mu_i)$. Since we also have $\mathrm{dom}(\nu) \subseteq S_i$, we obtain $\mathrm{dom}(\nu) \subseteq S_i \cap \mathrm{dom}(\mu_i)$ and hence we obtain finally that $\mathrm{graph}(\nu) \subseteq \mathrm{graph}(\lhd_{S_i} \mu_i)$, or that $\nu \sqsubseteq \lhd_{S_i} \mu_i$, whenever $i_0 \le i$, as required. ∎

Notice that either part of this result implies that if we fix the set S, then the map $(X \to Y) \to (X \to Y)$ defined by $\mu \mapsto \lhd_S \mu$ is Scott continuous, and it is this map which is normally understood in the context of domain restriction within VDM♣.

3.4 Non-Scott-Continuous Operators

3.4.1 The Domain Removal Operator, $\ntrianglelefteq$

Given $\mu \in (X \to Y)$ and an element S of $\mathcal{P}(X)$, we define the *removal from μ of S* to be the partial function in $(X \to Y)$, denoted by $\ntrianglelefteq_S \mu$, which satisfies: (i) $\mathrm{dom}(\ntrianglelefteq_S \mu) = \mathrm{dom}(\mu) \setminus S$, and (ii) $\ntrianglelefteq_S \mu$ coincides with μ on $\mathrm{dom}(\mu) \setminus S$.

Thus, we now obtain a mapping $\ntrianglelefteq : \mathcal{P}(X) \times (X \to Y) \to (X \to Y)$ defined by $\ntrianglelefteq (S, \mu) = \ntrianglelefteq_S \mu$. Moreover, if S_1 and S_2 are elements of $\mathcal{P}(X)$ and $\mu \in (X \to Y)$, then $(\mathrm{dom}(\mu) \setminus S_2) \setminus S_1 = \mathrm{dom}(\mu) \setminus (S_1 \cup S_2)$. Thus, (ii) of Definition 3.6 is satisfied, and (i) and (iii) also, relative to the monoid $(\mathcal{P}(X), \cup, \emptyset)$. Thus, algebraically, the mapping $\ntrianglelefteq$ determines an action of the monoid $(\mathcal{P}(X), \cup, \emptyset)$ on $(X \to Y)$. However, this mapping clearly cannot be continuous when the Scott topology is placed on $\mathcal{P}(X)$ and on $(X \to Y)$, since for fixed μ the map $\mathcal{P}(X) \to (X \to Y)$: $S \mapsto \ntrianglelefteq (S, \mu)$ is not even monotone, although it is clearly anti-monotone.

However, we note that for any $S \in \mathcal{P}(X)$ and any $\mu \in (X \to Y)$, we have the identity $\triangleleft\!\!\!-(S, \mu) = \lhd(X \setminus S, \mu) = \lhd(\text{comp}(S), \mu)$, so that $\triangleleft\!\!\!-_S\, \mu = \lhd_{X \setminus S}\, \mu$. Since comp is an isomorphism of topological monoids by Proposition 3.5, it transforms the action of $\triangleleft\!\!\!-$ into an action of $\lhd$, and we immediately obtain from Theorem 3.10 the following result.

3.11 Theorem Suppose that $\mathcal{P}(X)$ is endowed with the Cantor topology and that $(X \to Y)$ is endowed with the Scott topology. Then the mapping $\triangleleft\!\!\!- : \mathcal{P}(X) \times (X \to Y) \to (X \to Y)$ defined by $\triangleleft\!\!\!- (S, \mu) = \triangleleft\!\!\!-_S\, \mu$ determines an action of the topological monoid $(\mathcal{P}(X), \cup, \emptyset)$ on the domain $(X \to Y)$.

Thus, for any fixed S, the mapping $(X \to Y) \to (X \to Y) : \mu \mapsto \triangleleft\!\!\!-_S\, \mu$ is Scott continuous, and it is this map which is normally understood in the context of domain removal within VDM$^\clubsuit$. Nevertheless, we have decided to discuss $\triangleleft\!\!\!-$ in this subsection, rather than in the previous one, because, for any fixed μ, the mapping $\mathcal{P}(X) \to (X \to Y)$: $S \mapsto \triangleleft\!\!\!-(S, \mu)$ is not continuous when $\mathcal{P}(X)$ and $(X \to Y)$ both carry the Scott topology, as already noted, although it is when $\mathcal{P}(X)$ carries the Cantor topology.

3.4.2 The Override Operator, †

Given $\mu, \nu \in (X \to Y)$, we define the partial map $\mu \dagger \nu \in (X \to Y)$, called the *override of* μ *by* ν, as follows:

$$(\mu \dagger \nu)(x) = \begin{cases} \nu(x) & \text{if } x \in \text{dom}(\nu), \\ \mu(x) & \text{if } x \in \text{dom}(\mu) \setminus \text{dom}(\nu). \end{cases}$$

Thus, we obtain a mapping $\dagger : (X \to Y) \times (X \to Y) \to (X \to Y)$ defined by $\dagger(\mu, \nu) = \mu \dagger \nu$. Fixing the second argument ν, the mapping $\mu \mapsto \mu \dagger \nu$ can easily be seen to be Scott continuous by the methods we have been using thus far, and we omit the details. However, if we fix the first argument μ, and consider the mapping $\nu \mapsto \mu \dagger \nu$, it is easy to see that this mapping is not monotonic and hence is not Scott continuous. This has the consequence that † is not Scott continuous on $(X \to Y) \times (X \to Y)$. In fact, † does have certain continuity properties involving both the Cantor and Scott topologies which become apparent when one considers the canonical decomposition of $\mu \dagger \nu$ given below. Nevertheless, a satisfactory treatment of override appears to require a topology which suitably refines both the Scott and Cantor topologies, and we intend to introduce a satisfactory candidate for this shortly. However, before doing this we first establish the following result.

3.12 Proposition In the Scott topologies on $(X \to Y)$ and on $\mathcal{P}(X)$, the mapping dom : $(X \to Y) \to \mathcal{P}(X)$ defined by $\mu \mapsto \text{dom}(\mu)$ is continuous.

Proof. If $\mu_1 \sqsubseteq \mu_2$, that is, $\text{graph}(\mu_1) \subseteq \text{graph}(\mu_2)$, then $\text{dom}(\mu_1) \subseteq \text{dom}(\mu_2)$ and so dom is monotonic. Suppose that $A = \{\mu_\alpha; \alpha \in I\}$ is a directed set in $(X \to Y)$, and let $\mu = \sup_{\alpha \in I} \mu_\alpha$. We must show that $\text{dom}(\mu) = \text{dom}(\sup_{\alpha \in I} \mu_\alpha) = \bigcup_{\alpha \in I} \text{dom}(\mu_\alpha)$. But μ is the partial function determined by the union of the graphs of the partial functions μ_α, and so the required equality is easily established, and dom is continuous. ■

However, dom is not continuous if the Scott topology on $\mathcal{P}(X)$ is replaced by the Cantor topology. To see this, take Y to be the set of natural numbers, and take a net A_i which converges to A, say, in the Scott topology on $\mathcal{P}(X)$ but does not converge in the Cantor topology. Define $\mu_i \in (X \to Y)$ by setting $\mu_i(x)$ to be equal to 1 for all $x \in A_i$ and is otherwise undefined. Similarly, define $\mu \in (X \to Y)$ by setting $\mu(x)$ to be equal to 1 for all $x \in A$ and is otherwise undefined. Then $\mathrm{dom}(\mu_i) = A_i \not\to A = \mathrm{dom}(\mu)$ in the Cantor topology. However, it is a simple application of Proposition 3.1 and of the ideas used in the proof of (2) of Theorem 3.10 to see that $\mu_i \to \mu$ in the Scott topology on $(X \to Y)$ so that dom is not continuous. This fact is unfortunate and has significant bearing on subsequent developments.

Using the operators we have introduced so far, we can represent $\mu \dagger \nu$ by means of the equality $\mu \dagger \nu = \lessdot_{\mathrm{dom}(\nu)} \mu \sqcup \nu$. This representation allows us to canonically decompose $\mu \dagger \nu$ into a composite of three mappings, in the following way.
(1) The first of the factors is the mapping $(X \to Y) \times (X \to Y) \to \mathcal{P}(X) \times (X \to Y) \times (X \to Y) : (\mu, \nu) \mapsto (\mathrm{dom}(\nu), \mu, \nu)$. Up to a reordering of the components, this mapping is the product $[\mathrm{dom}, \mathrm{Id}] \times \mathrm{Id}$, where Id denotes the identity map.
(2) The second factor is the mapping $\mathcal{P}(X) \times (X \to Y) \times (X \to Y) \to (X \to Y) \times (X \to Y) : (\mathrm{dom}(\nu), \mu, \nu) \mapsto (\lessdot_{\mathrm{dom}(\nu)} \mu, \nu)$, and is the product $\lessdot \times \mathrm{Id}$.
(3) The third factor is the mapping $(X \to Y) \times (X \to Y) \to (X \to Y) : (\lessdot_{\mathrm{dom}(\nu)} \mu, \nu) \mapsto \lessdot_{\mathrm{dom}(\nu)} \mu \sqcup \nu$ and is the mapping $\sqcup$.

Thus, in addition to the comments already made about the topology we wish to place on $(X \to Y)$, it is clear that if such a topology makes each of the mappings above continuous, then it will make the override operator continuous. If we give $(X \to Y)$ the Scott topology, then the first of these mappings is continuous provided $\mathcal{P}(X)$ carries the Scott topology by Proposition 3.12, but not if $\mathcal{P}(X)$ carries the Cantor topology. On the other hand, the second of the factors in the decomposition above is continuous if $\mathcal{P}(X)$ carries the Cantor topology by Theorem 3.11, but not if it carries the Scott topology. The way forward appears to be to provide $(X \to Y)$ with a suitable topology which makes both dom and $\lessdot$ continuous when $\mathcal{P}(X)$ carries the Cantor topology.

One possible candidate for a suitable topology on $(X \to Y)$ which springs to mind is the Lawson topology, much studied in the context of complete and continuous lattices, see [2]. As already noted, this topology is the common refinement of the Scott and lower topologies and therefore has some computational significance. Moreover, it is compact Hausdorff, at least in the case of continuous lattices, see [2, III.1.10]. To define this topology, one takes as basic open sets the sets $U \backslash \uparrow F$, where U is Scott open, F is a finite set and, of course, $\uparrow F = \{y; f \sqsubseteq y \text{ for some } f \in F\}$. As a topology on $(X \to Y)$, the Lawson topology appears, unfortunately, not to be satisfactory for our purposes. For one thing, it turns out that dom is not continuous in this topology when $\mathcal{P}(X)$ has the Cantor topology, as we will see later. For another, related, reason it does not seem to be possible to formulate easily in these terms when a partial function μ is undefined at a point x, say. To say this, one would like to say $\mu \in U \backslash \uparrow \{[x \to y]; y \in Y\}$, where U is some Scott open set and $[x \to y]$ denotes the maplet, or partial function, whose value at x is y and is otherwise undefined. Unfortunately, the set $U \backslash \uparrow \{[x \to y]; y \in Y\}$ is not necessarily open in the Lawson topology if Y is infinite.

To solve this problem, we define a topology by means of convergence classes as described in Section 2. Thus, let $\mathcal{C}$ denote the set of all pairs (μ_i, μ), where μ_i is a net in $(X \to Y)$ and μ is an element of $(X \to Y)$, which satisfy the following condition: $(\mu_i, \mu) \in \mathcal{C}$ iff
(i) whenever $x \in \mathrm{dom}(\mu)$, eventually $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$, and
(ii) whenever $x \notin \mathrm{dom}(\mu)$, eventually $x \notin \mathrm{dom}(\mu_i)$.

Thus, μ_i converges $(\mathcal{C})$ to μ or $\lim_i \mu_i \equiv \mu$ $(\mathcal{C})$ iff the conditions (i) and (ii) are satisfied.

We show next that the conditions above define a convergence class.

3.13 Theorem The condition μ_i converges $(\mathcal{C})$ to μ or $\lim_i \mu_i \equiv \mu$ $(\mathcal{C})$ iff:
(i) whenever $x \in \mathrm{dom}(\mu)$, eventually $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$, and
(ii) whenever $x \notin \mathrm{dom}(\mu)$, eventually $x \notin \mathrm{dom}(\mu_i)$
determines a convergence class on $(X \to Y)$.

Proof. We must check that the four conditions of Definition 2.11 are satisfied.

(a) If μ_i is a constant net with value μ, it is clear that μ_i converges $(\mathcal{C})$ to μ.

(b) Suppose that $(\mu_i)_{i \in I}$ converges $(\mathcal{C})$ to μ and that $(\nu_j)_{j \in J}$ is a subnet of $(\mu_i)_{i \in I}$ determined by the function $\varphi : J \to I$. Thus, for each $j \in J$, we have $\nu_j = \mu_{\varphi(j)}$, and for each $i_0 \in I$ there is $j_0 \in J$ such that whenever $p \geq j_0$ in J we have $\varphi(p) \geq i_0$ in I. Suppose that $x \in \mathrm{dom}(\mu)$. Then there is $i_0 \in I$ such that $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$ whenever $i \geq i_0$. Choose the corresponding j_0 as above, and suppose that $p \geq j_0$. Then $\varphi(p) \geq i_0$. Thus, $\mathrm{graph}(\nu_p) = \mathrm{graph}(\mu_{\varphi(p)})$. But $\varphi(p) \geq i_0$, and therefore $(x, \mu(x)) \in \mathrm{graph}(\mu_{\varphi(p)})$. Hence, $(x, \mu(x)) \in \mathrm{graph}(\nu_p)$ and so eventually $(x, \mu(x)) \in \mathrm{graph}(\nu_j)$.

Now suppose that $x \notin \mathrm{dom}(\mu)$. Then there is $i_0 \in I$ such that whenever $i \geq i_0$, we have $x \notin \mathrm{dom}(\mu_i)$. Choose j_0 as above in the properties defining a subnet, and suppose that $p \geq j_0$. Then $\varphi(p) \geq i_0$. Thus, $\mathrm{dom}(\nu_p) = \mathrm{dom}(\mu_{\varphi(p)})$ and hence $x \notin \mathrm{dom}(\nu_p)$. Thus, eventually $x \notin \mathrm{dom}(\nu_j)$, as required. Consequently, ν_j converges $(\mathcal{C})$ to μ.

(c) Suppose that $(\mu_i)_{i \in I}$ does not converge $(\mathcal{C})$ to μ. Then one or other of the defining conditions for convergence $(\mathcal{C})$ is violated. Suppose that the first is violated. Thus, there is an $x \in \mathrm{dom}(\mu)$ such that $(x, \mu(x))$ is frequently not in $\mathrm{graph}(\mu_i)$. In other words, there is a cofinal subset J of I such that $(x, \mu(x)) \notin \mathrm{graph}(\mu_i)$ whenever $i \in J$. Clearly $(\mu_i)_{i \in J}$ determines a subnet of $(\mu_i)_{i \in I}$, in the usual way, no subnet of which converges $(\mathcal{C})$ to μ.

If the second defining condition of convergence $(\mathcal{C})$ is violated, we proceed similarly, and hence the third requirement, (c), is satisfied.

(d) Suppose that I is a directed set, let J_m be a directed set for each $m \in I$ and let F be the product directed set $I \times \prod_{m \in I} J_m$. Let $r : F \to I \times_I \bigcup_{m \in I} J_m$ be defined by $r(m, f) = (m, f(m))$. Now suppose that $\mu(m, n) \in (X \to Y)$ for all $m \in I, n \in J_m$, and suppose that $\lim_m \lim_n \mu(m, n) \equiv \nu$ $(\mathcal{C})$, where $\nu \in (X \to Y)$. We must show that $\mu \circ r$ converges $(\mathcal{C})$ to ν.

Let $x \in \mathrm{dom}(\nu)$. We must find $(m, f) \in F$ such that if $(p, g) \geq (m, f)$ in F, then $(x, \nu(x)) \in \mathrm{graph}(\mu \circ r(p, g))$. Choose $m \in I$ so that $(x, \nu(x)) \in \mathrm{graph}(\lim_n \mu(p, n))$ for each $p \geq m$. Now, $(x, \nu(x)) \in \mathrm{graph}(\lim_n \mu(p, n))$ means that $x \in \mathrm{dom}(\lim_n \mu(p, n))$ and $(\lim_n \mu(p, n))(x) = \nu(x)$, and moreover $\mu(p, n)$ converges $(\mathcal{C})$ to $\lim_n \mu(p, n)$ (for fixed p, and varying n). Therefore, for each $p \geq m$ we can choose $f(p) \in J_p$ such that $(x, \lim_n \mu(p, n)(x)) \in \mathrm{graph}(\mu(p, n))$ whenever $n \geq f(p)$ in J_p. Thus, $(x, \nu(x)) \in \mathrm{graph}(\mu(p, n))$ for all $n \geq f(p)$ in J_p. If p is a member of I which does not follow m in I (i.e. $m \not\leq p$), then let $f(p)$ be an arbitrary member of J_p, so that f is now everywhere defined. Now, if $(p, g) \geq (m, f)$ in F, then $p \geq m$ so that $(x, \nu(x)) \in \mathrm{graph}(\lim_n \mu(p, n))$ and since $g(p) \geq f(p)$ we have $(\lim_n \mu(p, n))(x) = \nu(x)$. Hence, $(x, \nu(x)) \in \mathrm{graph}(\mu(p, g(p)))$, that is, $(x, \nu(x)) \in \mathrm{graph}(\mu \circ r(p, g))$. Thus, $(x, \nu(x))$ is eventually in $\mathrm{graph}(\mu \circ r(p, g))$, as required.

Now suppose that $x \notin \mathrm{dom}(\nu)$. Again, we must find $(m, f) \in F$ such that if $(p, g) \geq (m, f)$, then $x \notin \mathrm{dom}(\mu \circ r(p, g))$. Choose $m \in I$ such that $x \notin \mathrm{dom}(\lim_n \mu(p, n))$ for each $p \geq m$. Since $\mu(p, n)$ converges $(\mathcal{C})$ to $\lim_n \mu(p, n)$ (for fixed p, and varying n), for each $p \geq m$ we can choose $f(p) \in J_p$ such that $x \notin \mathrm{dom}(\mu(p, n))$ whenever $n \geq f(p)$. If p is a member of I which does not follow m, then again let $f(p)$ be arbitrary in J_p. Now, if $(p, g) \geq (m, f)$, then $p \geq m$ and hence $x \notin \mathrm{dom}(\lim_n \mu(p, n))$ and hence $x \notin \mathrm{dom}(\mu(p, g(p))$ since $g(p) \geq f(p)$. So, $x \notin \mathrm{dom}(\mu \circ r(p, g))$, as required.

Thus, $\mu \circ r$ converges $(\mathcal{C})$ to ν, and the proof is complete. ■

This theorem results in a topology on $(X \to Y)$ which we will refer to as the *strong Cantor topology*, and the following remarks will explain why this name has been chosen.

3.14 Remark
(1) The conditions specifying convergence $(\mathcal{C})$ in Theorem 3.13 are easily seen to be equivalent to the following:
(a) whenever $(x, y) \in \mathrm{graph}(\mu)$, eventually $(x, y) \in \mathrm{graph}(\mu_i)$,
(b) whenever $x \notin \mathrm{dom}(\mu)$, eventually $x \notin \mathrm{dom}(\mu_i)$, and
(c) whenever $x \in \mathrm{dom}(\mu)$ and $(x, y) \notin \mathrm{graph}(\mu)$, then eventually $x \in \mathrm{dom}(\mu_i)$ and $(x, y) \notin \mathrm{graph}(\mu_i)$.
(2) Any partial function in $(X \to Y)$ may be identified with its graph and hence with a subset of $X \times Y$, so that we have a natural inclusion $(X \to Y) \subseteq \mathcal{P}(X \times Y)$ inside the power set of $X \times Y$. Thus, any convergence class respectively topology on $\mathcal{P}(X \times Y)$ induces a convergence class respectively topology on $(X \to Y)$. There are three such related to the present discussion:
(i) Convergence of a net A_i to A in $\mathcal{P}(X \times Y)$ specified by "whenever $(x, y) \in A$ we eventually have $(x, y) \in A_i$". By Example 3.3 and Proposition 3.1 this induces the Scott topology on $\mathcal{P}(X \times Y)$ and, as a subspace, induces the Scott topology on $(X \to Y)$, see Proposition 3.7.
(ii) Convergence of a net A_i to A in $\mathcal{P}(X \times Y)$ specified by "whenever $(x, y) \in A$ we eventually have $(x, y) \in A_i$, and whenever $(x, y) \notin A$ we eventually have $(x, y) \notin A_i$". By Example 3.3, we obtain the Cantor topology on $\mathcal{P}(X \times Y)$. We shall refer to the induced topology on $(X \to Y)$ in this case as the *Cantor topology* on $(X \to Y)$. Of course, as far as $(X \to Y)$ is concerned, a net μ_i converges to μ in the Cantor topology iff whenever $(x, y) \in \mathrm{graph}(\mu)$ we eventually have $(x, y) \in \mathrm{graph}(\mu_i)$, and whenever $(x, y) \notin \mathrm{graph}(\mu)$ we eventually have $(x, y) \notin \mathrm{graph}(\mu_i)$.
(iii) Thinking of a subset A of $X \times Y$ as a relation from X to Y, we define the *domain* of A, $\mathrm{dom}(A)$, to be the set $\{x \in X; (x, y) \in A \text{ for some } y \in Y\}$. We can therefore generalize the conditions stated in (1), and hence those stated in Theorem 3.13, by specifying convergence of A_i to A in $\mathcal{P}(X \times Y)$ to mean:
(a)$'$ whenever $(x, y) \in A$, eventually $(x, y) \in A_i$,
(b)$'$ whenever $x \notin \mathrm{dom}(A)$, eventually $x \notin \mathrm{dom}(A_i)$, and
(c)$'$ whenever $x \in \mathrm{dom}(A)$ and $(x, y) \notin A$, then eventually $x \in \mathrm{dom}(A_i)$ and $(x, y) \notin A_i$.
Following the proof of Theorem 3.13, it can be shown that this notion of convergence also defines a convergence class. Of course, the topology it induces on $(X \to Y)$ is the strong Cantor topology.
(3) It follows easily from the previous remark (1) that the convergence just defined, namely (2) (iii), implies that in (2) (ii) so that the strong Cantor topology is a refinement of the Cantor topology, and hence the name "strong Cantor topology".

We next collect together some basic facts about the topologies we have been discussing.

3.15 Proposition The following facts hold.
(1) The strong Cantor topology is a refinement of the Cantor topology which in turn is a refinement of the Scott topology.
(2) The set $(X \to Y)$ is closed in $\mathcal{P}(X \times Y)$ in each of the three topologies on $\mathcal{P}(X \times Y)$ discussed in Remark 3.14.

(3) The space $(X \to Y)$ is compact and T_0 in the Scott topology.
(4) The space $(X \to Y)$ is compact Hausdorff in the Cantor topology.
(5) The space $(X \to Y)$ is Hausdorff in the strong Cantor topology and is compact iff the Cantor and strong Cantor topologies coincide.
(6) In general, the Cantor and strong Cantor topologies do not coincide, and therefore the strong Cantor topology is not generally compact.
(7) The strong Cantor topology is not trivial, that is, it is not the discrete topology.
(8) The space Y^X of all total functions mapping X into Y is not a closed subset of $(X \to Y)$ in the Scott and Cantor topologies, but is closed in the strong Cantor topology. In each of the three topologies in question, the induced topology on Y^X is not trivial, that is, is not discrete.
(9) The strong Cantor and Cantor topologies coincide on the set Y^X of all total functions in $(X \to Y)$.

Proof. (1) By Remark 3.14, the net convergence condition describing the strong Cantor topology is more restrictive than that describing the Cantor topology which in turn is more restrictive than that describing the Scott topology, and this observation suffices.
(2) Let A_i be a net of sets in $\mathcal{P}(X \times Y)$, each element of which is the graph of a partial function in $(X \to Y)$ and suppose that $A \in \mathcal{P}(X \times Y)$. By Remark 3.14, if A_i converges to A in any of the topologies in question, at least the following condition holds: whenever $(x, y) \in A$, eventually $(x, y) \in A_i$. Thus, if (x, y') is also an element of A, then eventually it too is an element of A_i. But then it is immediate that $y = y'$ and hence that A is the graph of a partial function in $(X \to Y)$, as required.
(3) This is a well-known general fact about the Scott topology.
(4) The space $\mathcal{P}(X \times Y)$ is compact and Hausdorff in the Cantor topology. Hence, $(X \to Y)$ is also Hausdorff and, being closed by (2), is also compact.
(5) Since the strong Cantor topology is a refinement of the Cantor topology, it also is Hausdorff and so therefore is $(X \to Y)$ in this topology. If the Cantor and strong Cantor topologies agree, then obviously $(X \to Y)$ is also compact in the strong Cantor topology, by (2) and (4). On the other hand, if $(X \to Y)$ is compact in the strong Cantor topology, then the identity map regarded as a map from this space to $(X \to Y)$ with the Cantor topology is a one-to-one continuous mapping from a compact space onto a Hausdorff space, and therefore is a homeomorphism.
(6) Consider the partial functions $(\mathbb{N} \to \mathbb{N})$ from the set of natural numbers to itself. Define μ by setting $\mu(x) = 1$ if x is even, and taking μ to be undefined otherwise. For each $n \in \mathbb{N}$, define μ_n as follows: $\mu_n(x) = 1$ if x is even and $x \leq n$; $\mu_n(x) = n$ if x is odd and $x \leq n$; $\mu_n(x)$ is undefined if $x > n$. Then μ_n does not converge to μ in the strong Cantor topology because $\mathrm{dom}(\mu_n)$ does not converge to $\mathrm{dom}(\mu)$ in the Cantor topology on $\mathcal{P}(X)$, see Proposition 3.16 below. Yet μ_n does converge to μ in the Cantor topology. Thus, the Cantor and strong Cantor topologies are different in this case, and hence by the previous result (5) the space $(X \to Y)$ is not compact in the latter topology.
(7) Again consider $(\mathbb{N} \to \mathbb{N})$. Take μ as defined in (6). This time, for each natural number n, define μ_n by setting $\mu_n(x) = 1$ if x is even and $x \leq n$, and taking μ_n to be undefined otherwise. Then the sequence $(\mu_n)_n$ is not eventually constant and yet converges to μ in the strong Cantor topology.
(8) Take μ and μ_n as defined in (6) except that we now set $\mu_n(x) = 0$ if $x > n$, so that each of the μ_n is a total function. Again, μ_n does not converge to μ in the strong Cantor topology, but does so in the Scott and Cantor topologies. Since μ is not total, it follows that Y^X is not

closed in $(X \to Y)$ in the Scott and Cantor topologies. On the other hand, if μ_n converges to μ in the strong Cantor topology and each of the μ_n is total, then so is μ by Proposition 3.16, and hence Y^X is closed in this case. Finally, take $(\mathbb{N} \to \mathbb{N})$ again, define μ by setting $\mu(x) = 1$ for all x and define μ_n by setting $\mu_n(x) = 1$ if $0 \leq x \leq n$ and setting $\mu_n(x) = 0$ otherwise; so that μ and each μ_n are total. Then $\mu_n \to \mu$ in the strong Cantor topology, and yet $(\mu_n)_n$ is not eventually constant, and so the induced topology on Y^X is not discrete.
(9) Let μ_i and μ be total functions in $(X \to Y)$. If $\mu_i \to \mu$ in the strong Cantor topology, then certainly $\mu_i \to \mu$ in the Cantor topology by (1). Conversely, suppose that $\mu_i \to \mu$ in the Cantor topology. Let $x \in \text{dom}(\mu)$. Then $(x, \mu(x)) \in \text{graph}(\mu)$ and so, by our current assumption, we have that $(x, \mu(x)) \in \text{graph}(\mu_i)$ eventually. The second convergence condition defining the strong Cantor topology is trivially satisfied in this case, and the result follows. ■

Notice that (9) of Proposition 3.15 applies in particular when Y is the two-element set. Therefore, the strong Cantor and Cantor topologies coincide on the power set $\mathcal{P}(X)$. We will, however, persist in what follows in referring to the Cantor topology on $\mathcal{P}(X)$, rather than using the all-embracing term "strong Cantor topology".

We now proceed with our treatment of the override operator armed with the results that we have established at our disposal. We start with the following simple, but important, fact.

3.16 Proposition If μ_i converges to μ in the strong Cantor topology, then $\text{dom}(\mu_i)$ converges to $\text{dom}(\mu)$ in the Cantor topology on $\mathcal{P}(X)$. Hence, the map dom is continuous when $(X \to Y)$ is endowed with the strong Cantor topology and $\mathcal{P}(X)$ is endowed with the Cantor topology.

Proof. Suppose that $\mu_i \to \mu$ in the strong Cantor topology. Let $x \in \text{dom}(\mu)$. Then eventually $(x, \mu(x)) \in \text{graph}(\mu_i)$ and hence eventually $x \in \text{dom}(\mu_i)$. On the other hand, if $x \notin \text{dom}(\mu)$, then eventually $x \notin \text{dom}(\mu_i)$. Thus, $\text{dom}(\mu_i) \to \text{dom}(\mu)$ in the Cantor topology on $\mathcal{P}(X)$, as required. ■

3.17 Remark

(1) The examples in (6) and (8) of Proposition 3.15 show that the function dom is not continuous when $(X \to Y)$ and $\mathcal{P}(X)$ both have the Cantor topologies, so that the Cantor topology on $(X \to Y)$ is not entirely satisfactory. Since the Lawson topology on $\mathcal{P}(X \times Y)$ coincides with the Cantor topology, the Lawson topology is also not entirely satisfactory in this context.
(2) These comments raise the general question "Just what is a reasonable notion of convergence in $(X \to Y)$?", the answer depending of course on the ultimate applications. In subjects like functional analysis, for example, convergence is often uniform ($\sup_x ||f(x) - f_n(x)|| \to 0$ as $n \to \infty$ in some norm $|| \cdot ||$) or uniform convergence on compacta etc. These notions are intuitively very reasonable and do indeed capture the notion of the function f_n tending towards the function f. However, the price one pays for this is relatively few convergent sequences (or nets), hence relatively many open sets, hence the underlying topology is highly non-compact, usually. By comparison, convergence in $(X \to Y)$ can be quite bizarre as shown by some of the examples given in the proof of Proposition 3.15. In fact, convergence of $\text{dom}(\mu_i)$ to $\text{dom}(\mu)$ as part of one's definition is quite natural and, intuitively, convergence in the strong Cantor topology is probably the most reasonable. However, again the price to be paid is non-compactness in general.

Notwithstanding the remarks just made about non-compactness, it will become apparent later that the following result shows that the strong Cantor topology is in many ways the best possible choice of topology to impose on $(X \to Y)$.

3.18 Proposition The strong Cantor topology is the smallest topology on $(X \to Y)$ which refines both the Scott topology and the Lawson topology and in which the function dom is continuous when $\mathcal{P}(X)$ is endowed with the Cantor (or Lawson) topology.

Proof. By Proposition 3.15, the strong Cantor topology certainly refines both the Scott topology and the Lawson topology, and by Proposition 3.16 satisfies the continuity requirement concerning dom.

Now suppose $\mathcal{T}$ is any topology refining both the topologies mentioned and such that dom is continuous. Let $\mu_i \to \mu$ in $\mathcal{T}$. Then $\mu_i \to \mu$ in the Scott topology. Let $x \in \mathrm{dom}(\mu)$. Then $(x, \mu(x)) \in \mathrm{graph}(\mu)$. Hence, by Proposition 3.7, $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$ eventually. On the other hand, $\mathrm{dom}(\mu_i) \to \mathrm{dom}(\mu)$ in the Cantor topology on $\mathcal{P}(X)$. Hence, if $x \notin \mathrm{dom}(\mu)$, then eventually $x \notin \mathrm{dom}(\mu_i)$. Thus, $\mu_i \to \mu$ in the strong Cantor topology, as required. ■

3.19 Theorem The mapping $\lhd : \mathcal{P}(X) \times (X \to Y) \to (X \to Y)$ defined by $\lhd(S, \mu) = \lhd_S \mu$ is continuous when $\mathcal{P}(X)$ is endowed with the Cantor topology and $(X \to Y)$ is endowed with either:
(i) the strong Cantor topology, or
(ii) the Cantor topology.

Proof. We content ourselves by proving the result referring to (i) and leave the details of (ii), which are similar, to the reader.

Thus, suppose that $(S_i, \mu_i) \to (S, \mu)$ in $\mathcal{P}(X) \times (X \to Y)$. Thus, $S_i \to S$ in the Cantor topology on $\mathcal{P}(X)$ and $\mu_i \to \mu$ in the strong Cantor topology on $(X \to Y)$. We must show that $\lhd_{S_i} \mu_i \to \lhd_S \mu$ in the strong Cantor topology.

Suppose that $x \in \mathrm{dom}(\lhd_S \mu)$. Thus, $x \in S$ and $x \in \mathrm{dom}(\mu)$. Since $S_i \to S$, we eventually have $x \in S_i$; since $\mu_i \to \mu$, we eventually have $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$. Using the directedness of the index set of the net in question, we can choose the index i_0 to get these statements holding simultaneously beyond i_0. Thus, beyond i_0 we have $x \in S_i \cap \mathrm{dom}(\mu_i)$ and $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$. Hence, beyond i_0 we have $\mu_i(x) = \mu(x)$ and hence we have $\lhd_{S_i} \mu_i(x) = \lhd_S \mu(x)$. Thus, $(x, \lhd_S \mu(x)) \in \mathrm{graph}(\lhd_{S_i} \mu_i)$ beyond i_0 and, hence, eventually.

Now suppose that $x \notin \mathrm{dom}(\lhd_S \mu) = S \cap \mathrm{dom}(\mu)$.
Case 1: $x \notin S$. Since $S_i \to S$, eventually $x \notin S_i$ and hence eventually $x \notin S_i \cap \mathrm{dom}(\mu_i)$ so that eventually $x \notin \mathrm{dom}(\lhd_{S_i} \mu_i)$.
Case 2: $x \notin \mathrm{dom}(\mu)$. Since $\mu_i \to \mu$, eventually $x \notin \mathrm{dom}(\mu_i)$, and hence eventually $x \notin S_i \cap \mathrm{dom}(\mu_i)$, that is, eventually $x \notin \mathrm{dom}(\lhd_{S_i} \mu_i)$.

This covers all cases, and so now we see that $\lhd_{S_i} \mu_i \to \lhd_S \mu$, as required. ■

Bearing in mind our earlier comments about comp transforming the action of $⩤$ into one of $\lhd$ and vice versa, we immediately obtain from Theorem 3.19 the following result.

3.20 Theorem The mapping $⩤ : \mathcal{P}(X) \times (X \to Y) \to (X \to Y)$ defined by $⩤(S, \mu) = ⩤_S \mu$ is continuous when $\mathcal{P}(X)$ is endowed with the Cantor topology and $(X \to Y)$ is endowed with either:
(i) the strong Cantor topology, or
(ii) the Cantor topology.

3.21 Theorem The mapping $\sqcup : (X \to Y) \times_{\sqcup} (X \to Y) \to (X \to Y)$ defined by $\sqcup(\mu, \nu) = \mu \sqcup \nu$ is continuous when $(X \to Y)$ is endowed with either:
(i) the strong Cantor topology, or
(ii) the Cantor topology.

Proof. Again, we prove just the first of these claims and leave the second to the reader.

Let (μ_i, ν_i) be a net converging in $(X \to Y) \times_{\sqcup} (X \to Y)$ to (μ, ν) relative to the strong Cantor topology. We must show that $\mu_i \sqcup \nu_i$ converges in this topology to $\mu \sqcup \nu$. Now $\mu_i \to \mu$ and $\nu_i \to \nu$. Suppose that $x \in \mathrm{dom}(\mu \sqcup \nu) = \mathrm{dom}(\mu) \cup \mathrm{dom}(\nu)$ and for sake of argument suppose that $x \in \mathrm{dom}(\mu)$. Then eventually $(x, \mu(x)) \in \mathrm{graph}(\mu_i)$ and hence eventually $(x, \mu(x)) \in \mathrm{graph}(\mu_i \sqcup \nu_i)$, since $(\mu_i \sqcup \nu_i)(x) = \mu_i(x) = \mu(x)$. Similarly, if $x \in \mathrm{dom}(\nu)$.

Now suppose that $x \notin \mathrm{dom}(\mu \sqcup \nu)$. Then $x \notin \mathrm{dom}(\mu)$ and $x \notin \mathrm{dom}(\nu)$. Thus, eventually we simultaneously get $x \notin \mathrm{dom}(\mu_i)$ and $x \notin \mathrm{dom}(\nu_i)$. Hence, eventually we get $x \notin \mathrm{dom}(\mu_i \sqcup \nu_i)$, as required. ∎

Recalling the canonical decomposition of the override operator and using the results just established we now obtain the following main result.

3.22 Theorem The mapping $\dagger : (X \to Y) \times (X \to Y) \to (X \to Y)$ defined by $\dagger(\mu, \nu) = \mu \dagger \nu$ is continuous in the strong Cantor topology.

4 Compactness of $(X \to Y)$

To close, we consider the compactness of the space $(X \to Y)$ in the strong Cantor topology. To do this, let $\perp$ be some object not in Y, let $Y_\perp$ denote the set $Y \cup \{\perp\}$, and let $\perp^X$ denote the constant map on X with value $\perp$.

4.1 Proposition For any set Y, the Cantor topology on Y^X coincides with the topology of pointwise convergence, where Y has the discrete topology.

Proof. Suppose that $f_i \to f$ in the Cantor topology on Y^X, and that $f(x) = y$. Then $(x, y) \in \mathrm{graph}(f)$ and hence eventually $(x, y) \in \mathrm{graph}(f_i)$. Thus, eventually $f_i(x) = y$ and so $f_i(x)$ converges to y in the discrete topology on Y. Conversely, suppose that $f_i \to f$ in the topology of pointwise convergence, where Y has the discrete topology. If $(x, y) \in \mathrm{graph}(f)$, then $f(x) = y$ and so eventually $f_i(x) = y$, that is, eventually $(x, y) \in \mathrm{graph}(f_i)$. Now suppose that $(x, y) \notin \mathrm{graph}(f)$. Then $f(x) \neq y$, and so eventually $f_i(x) \neq y$ and consequently we have eventually that $(x, y) \notin \mathrm{graph}(f_i)$. Hence, $f_i \to f$ in the Cantor topology on Y^X. ∎

Let $t : (X \to Y) \to Y_\perp$ be the mapping defined by $t(\mu) = \perp^X \dagger \mu$, and let p be the mapping $p : Y_\perp \to (X \to Y)$, where $p(f) = f'$ is the partial map f' obtained by the removal from the total map f of the set S of all those $x \in X$ such that $f(x) = \perp$. These mappings are bijections and each is the inverse of the other. Indeed, in the strong Cantor topology each is a homeomorphism as we see in the proof of the following result.

4.2 Proposition The space $(X \to Y)$ is compact in the strong Cantor topology if and only if Y is finite.

Proof. When endowed with the strong Cantor topology, both $(Y_\perp)^X$ and $(X \to Y)$ are subspaces of $(X \to Y_\perp)$ in the strong Cantor topology. Therefore, on fixing the first factor of $\dagger$ at $\perp^X$ in Theorem 3.22, we see that t is continuous in the strong Cantor topology.

The map p is also continuous in the strong Cantor topology, as we now show. Suppose that $f_i \to f$ in the strong Cantor topology on $(Y_\perp)^X$. Let $x \in \mathrm{dom}(f')$, so that $f'(x) = f(x) \neq \perp$. Since $f_i \to f$, we have $(x, f(x))$ eventually in $\mathrm{graph}(f_i)$. Thus, eventually $f_i(x) = f(x) \neq \perp$. So, eventually $f_i'(x) = f'(x) \neq \perp$, that is, eventually $(x, f'(x)) \in \mathrm{graph}(f_i')$. Now suppose that $x \notin \mathrm{dom}(f')$. This means that $f(x) = \perp$. Since $f_i \to f$, eventually $(x, f(x)) \in \mathrm{graph}(f_i)$ so that eventually $f_i(x) = f(x) = \perp$ and hence eventually $x \notin \mathrm{dom}(f_i')$. Thus, $f_i' \to f'$ in the strong Cantor topology on $(X \to Y)$, as required. Thus, t and p are homeomorphisms.

Suppose now that $(X \to Y)$ is compact in the strong Cantor topology. Then $(Y_\perp)^X$ is compact in this topology and hence is compact in the Cantor topology by (9) of Proposition 3.15. But then $Y_\perp$ is finite by Proposition 4.1 and Tychonoff's theorem, and therefore Y is finite. Conversely, if Y is finite, then $(Y_\perp)^X$ is compact in the Cantor topology and hence in the strong Cantor topology, and therefore $(X \to Y)$ is compact in the strong Cantor topology. ■

One can avoid the introduction of the element $\perp$, and we briefly consider the following alternative way of proceeding to obtain the previous result. To do this we must, however, choose a base point in Y. Having made this choice, u say, we then let u^X denote the constant function on X with value u.

Let $p : \mathcal{P}(X) \times Y^X \to (X \to Y)$ be defined by $p(S, f) = \lhd_S f$, noting that no confusion will be caused by use of the symbol p again. It follows immediately from Theorem 3.19 that p is continuous when Y^X and $(X \to Y)$ have the strong Cantor topology, and $\mathcal{P}(X)$ has the Cantor topology. Furthermore, there is a naturally defined section $s : (X \to Y) \to \mathcal{P}(X) \times Y^X$ of this mapping p, where $s(\mu) = (\mathrm{dom}(\mu), u^X \dagger \mu,)$, and s is clearly continuous by virtue of Proposition 3.16 and the proof of the continuity of t given above.

The mappings p and s determine an endomorphism $e = s \circ p$ of the product space $\mathcal{P}(X) \times Y^X$ to itself which is continuous in the topologies under discussion. Since s is a section of p, it follows that e is idempotent and so $e(S, f)$ is a fixed point of e for each pair (S, f). Letting $F = \mathrm{fix}(e)$ denote the set of all fixed points of e, we note that F is a closed set in $\mathcal{P}(X) \times Y^X$, as is readily checked. We further note the following facts which are established in [6], see also [8]: (i) $(S, f) \in F$ if and only if $f(x) = u$ for all $x \in X \setminus S$, and (ii) the restriction $p|_F$ of p to F is an inverse of s.

We can now give an alternative proof of Proposition 4.2, as follows. First, we note that $(X \to Y)$ and F are homeomorphic (under $p|_F$ or s) and F is closed. Therefore, if Y is finite, it is immediate from Proposition 4.1 that F is compact and hence that $(X \to Y)$ is compact. Conversely, if $(X \to Y)$ is compact, then F is compact. But $(X, f) \in F$ for each $f \in Y^X$. Therefore, the image of F under the projection on the second factor on the product $\mathcal{P}(X) \times Y^X$ is all of Y^X and hence the latter is compact. It follows that Y is finite.

The identification above of $(X \to Y)$ and $(Y_\perp)^X$ using t and p is the one customarily used except that we have expressed it in the language of VDM. However, t is not continuous in the Scott topology nor in the Cantor topology, and this is one of the reasons why the strong Cantor topology seems to be quite appropriate for carrying out functional calculus on spaces of partial functions. In addition it gives a smooth treatment of such spaces and their operators, as shown by the mechanics of the proofs above.

5 Conclusions and Further Work

The results of the first part of the paper give a very satisfactory development of convergence spaces and convergence classes in terms of both nets and filters. Moreover, we establish complete duality between the two theories, and present simple conditions under which a convergence space is topological. Specifically, a convergence space $(X, \mathcal{S})$ in net form is topological iff it is a convergence class in net form iff it satisfies conditions (c) and (d) of Definition 2.11. Similarly, a convergence space $(X, \mathcal{F})$ in filter form is topological iff it is a convergence class in filter form iff it satisfies conditions (c) and (d) of Definition 2.14.

The second part shows that in relation to VDM, the Scott topology is not satisfactory: certain of the standard basic operators encountered there are Scott continuous, others are not. Overcoming this has necessitated the introduction of (smallest possible) refinements of the Scott topology such as the strong Cantor topology and then all the basic operators considered are continuous, giving a satisfactory analysis.

Noting that "constructive" and "effective" are closely related concepts, and that "effective" and "continuous" are also closely related from the domain-theoretic point of view, it is of interest to investigate the effectiveness of the operators we have discussed within the topological framework of this paper. This objective is closely related to the programme being carried out in [6, 9, 10], where the operators we have considered (particularly the override) have been studied in [6, 9, 10] from the point of view of topos theory in order to view them constructively. Thus, one may view the present paper as taking a first step towards examining the effectiveness of the operators concerned by considering the possibility of constructive topology within VDM$^\clubsuit$ in the spirit of [10]. Indeed, one of our objectives here has been to provide a "convenient category" in which all the operations considered are automatically continuous, and the strong Cantor topology essentially does this. However, the full objective of contributing an appropriate notion of effectiveness within the framework of [10] is ongoing work of the authors and will be discussed elsewhere.

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